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Fuzzy fractional diffusion equation with time delay having power law and Mittag-Leffler kernel

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Abstract

In this article, we study a new type of mathematical model of the diffusion equation. This model has space and time-fractional derivative with both singular and non-singular derivative i.e., having power-law kernel and Mittag-Leffler kernel. This model is taken in a fuzzy environment. A delay term is also added to this model. First, we approximate the unknown fuzzy function as a truncated series expansion of shifted Chebyshev polynomials. We find an approximate formula for fractional derivatives and delay function. The main advantage of this method in applying this method is, the taken fuzzy fractional partial differential equation (FF-PDE) reduces into a system of a non-linear fuzzy algebraic equation. We solve this system by any method available in the literature and find the value of unknowns. We depict the validity and feasibility of this derived method by deriving numerical solution of particular cases of taken fuzzy model. We observe that this method is valid for a fuzzy fractional differential equation with delay term and has good accuracy as seen from the error tables.

1. Introduction

Fractional calculus is an old branch of mathematics that allow to extend the integer order differentiation and integration to the real or arbitrary order. The Gamma function play a crucial role in fractional order operator definitions. Researchers gave many types of fractional order derivatives and integrations. Some important are Caputo , Riemann-Liouville, Grunwald-Letnikov and Hadamard. In the initial phase of fractional calculus development, the power law kernel fractional operators are introduced. But these operators have their limitations in complex phenomenas. To overcome this problem and to describe the behavior of complex phenomenas fractional operators with non-singular kernel are introduced. Some of them are very popular now-days like Caputo-Fabrizio derivative and Atangana-Baleanu derivative. These derivatives have exponential kernel and Mittag-Leffler kernel respectively. The variable order derivatives in which the order is a function of one or more variable has been introduced with many applications. The fractional order differential equation are complex and to find out their solution is a tough task. Thus many numerical methods have been developed to solve fractional order ODEs and PDEs like as homotopy perturbation method [1], fractional differential transform method [2], Adomain decomposition method [3], predictor-corrector method [4] and operational matrix method based upon block pulse functions [5]. In daily life, many phenomenas and systems have such property that it can be depicted by deterministic model. In such models, parameters and functions attain value from an interval. Fuzzy modeling can easily describe the behavior of such models. The application of fuzzy modeling can be seen in many areas and nowadays it is progressing very much. There are many numerical methods dealing with fractional fuzzy models like as implicit finite difference [6], Laplace tranform method for solving FFDEs [7] and Adomain decomposition [8].

We see the many phenomena in our daily life that have diffusion process as so common. We are so much familiar with the process of diffusion. In this phenomena, a physical quantity like molecules moves from the higher concentration region to the lower concentration region. This process is describes by a PDE known as diffusion equation. In the addition of diffusion process many other processes also occurs simultaneously like as reaction and advection phenomena. In reaction process the molecules reacts with other physical substance while in advection represent the bulk motion of molecules with the fluid substance. The fractional diffusion equation has been derived in last few years to depict the anomalous diffusion process which behavior was unable to predict with integer order diffusion equation. The following equation represent the fractional form of advection reaction diffusion equation

$$\frac{\partial^{\delta}\Xi(\zeta,\eta)}{\partial\eta^{\delta}} = D\nabla^2\Xi(\zeta,\eta) - d\frac{\partial\Xi(\zeta,\eta)}{\partial\eta} + R_1(\Xi,\zeta,\eta). \tag{1}$$

The term $\nabla^2\Xi(\zeta,\eta)$ denotes the diffusion process with D as diffusion coefficient.

The article has been divided into 6 sections. Section 2 has some properties of fuzzy calculus that will be used in this article. The error bound has been given in section 3. The model has been described along with developed method has been included in section 4. The results and validation of method is given in section 5. The overall conclusion can be seen in last section.

2. Some basic concepts and definitions of fuzzy set and fuzzy calculus

In this section, the definition of fuzzy set, fuzzy number, fuzzy sum and differentiability on fuzzy environment is given. The fuzzy form of Caputo and Mittag-Lefler kernel fractional derivative is introduced here [9].

Definition 1: A fuzzy number is nothing but a mapping from the set of real numbers to the closed interval $[0, 1]$

$$\tilde{\sigma} : R \longrightarrow [0, 1]$$

which satisfy some certain properties given as follows

1. The property of upper semi continuity is satisfied by mapping $\tilde{\sigma}$.
2. Convexity
$$\tilde{\sigma}(\mu\zeta + (1-\mu)z) \leq \min\{\tilde{\sigma}(\zeta), \tilde{\sigma}(z)\}, \mu \in [0,1] \& \forall \zeta, z \in R. \tag{2}$$
3. There exists a point $z_0 \in R$ such that the mapping $\tilde{\sigma}$ satisfies $\tilde{\sigma}(z_0) = 1$.
4. If we define the support of mapping $\tilde{\sigma}$ by
$$supp(\tilde{\sigma}) = \{z \in R : \tilde{\sigma}(z) > 0\} \tag{3}$$
then this support must satisfy the property of compactness.

Definition 2: For a fuzzy number $\tilde{\sigma} \in R_F$, the s-level set is defined as follows with

$$[\tilde{\sigma}_s] = \begin{cases} \{z \in R : \tilde{\sigma}(z) \geq s\}, & s \in (0, 1] \\ closure(supp(\tilde{\sigma})), & s = 0. \end{cases} \tag{4}$$

Here, R_F stands for the set of all fuzzy numbers defined on set of real numbers.
From the equation 4, it can be seen that level sets are closed and bounded interval i.e., $[\tilde{\sigma}_(s)] = [\tilde{\sigma}^-(s), \tilde{\sigma}^+(s)]$.

Definition 3: The algebraic operations between two fuzzy numbers the fuzzy numbers $\tilde{\sigma}$ and $\tilde{w}$ are defined as

1. $\tilde{\sigma} \oplus \tilde{w} = (\tilde{\sigma}^- + \tilde{w}^-, \tilde{\sigma}^+ + \tilde{w}^+)$,
2. $\varLambda \odot \tilde{\sigma} = \begin{cases} (\varLambda\tilde{\sigma}^-, \varLambda\tilde{\sigma}^+), & \varLambda \geq 0, \\ (\varLambda\tilde{\sigma}^+, \varLambda\tilde{\sigma}^-), & \varLambda < 0. \end{cases}$

The symbols $\oplus$ and $\odot$ denote the addition and multiplication of fuzzy numbers.

Definition 4: The metric on space of fuzzy numbers is a mapping with domain $R_F \times R_F$ and range $R^+ \cup 0$. The distance between two fuzzy numbers is defined as [10, 11, 12, 13]

$$\varPsi(\tilde{\sigma}, \tilde{w}) = \sup_{s\in[0,1]} \max\{|\tilde{\sigma}^-(s) - \tilde{w}^-(s)|, |\tilde{\sigma}^+(s) - \tilde{w}^+(s)|\}. \tag{5}$$

The above distance function satisfies the properties as follows

1. $\Psi(\tilde{\sigma} \oplus \tilde{\zeta}, \tilde{w} \oplus \tilde{\zeta}) = \Psi(\tilde{\sigma}, \tilde{w})$, $\forall \tilde{\sigma}, \tilde{w}, \tilde{\zeta} \in R_F$.
2. $\Psi(a \odot \tilde{\sigma}, a \odot \tilde{w}) = |a|\Psi(\tilde{\sigma}, \tilde{w}), \forall a \in R, \tilde{\sigma}, \tilde{w} \in R_F$.
3. $\Psi(\tilde{\sigma} \oplus \tilde{w}, \tilde{\zeta} \oplus \tilde{z}) \leq \Psi(\tilde{\sigma}, \tilde{\zeta}) + \Psi(\tilde{w}, \tilde{z})$, $\forall \tilde{\sigma}, \tilde{w}, \tilde{\zeta}, \tilde{z} \in R_F$.

The above metric space satisfies the property of completeness.

Definition 5:[14, 15] The H-difference between two fuzzy numbers is defined with the help of addition operation. We say $\tilde{v}$ is the H-difference of another two fuzzy numbers $\tilde{w}$ and $\tilde{x}$ if

$$\tilde{w} = \tilde{x} \oplus \tilde{v} \tag{6}$$

It is denoted by by $\tilde{w} \ominus \tilde{x}$.

Definition 6: An another type of difference known as generalized difference (g-difference) is defined as

$$[\tilde{w} \ominus_g \tilde{z}]_r = closure\big\{ \cup_{s \geq r} \left([\tilde{w}]_s \ominus_{gH} [\tilde{z}]_s \right) \big\}, \tag{7}$$

where $[\tilde{w}]_s$ represent the s-level fuzzy set of $\tilde{w}$.

Definition 7 [16]: Now, we give the important definition of fuzzy differentiability which has been used throughout the article. Let us consider a function with domain open interval (c, d) and co-domain set of all fuzzy numbers on the set of real numbers. The fuzzy valued function $\tilde{\psi}$ is said to be differentiable at point z_0 if $\exists\ \tilde{\psi}'(z) \in R_F$ such that it satisfies

1. The H-differences $\tilde{\psi}(z_0 + k) \ominus \tilde{\psi}(z_0)$ and $\tilde{\psi}(z_0) \ominus \tilde{\psi}(z_0 - k)$ both must exist and

$$\lim_{k \to 0^+} \frac{\tilde{\psi}(z_0 + k \ominus \tilde{\psi}(z_0))}{k} = \frac{\tilde{\psi}(z_0) \ominus \tilde{\psi}(z_0 - k)}{k} = \tilde{\psi}'(z_0) \tag{8}$$

   This type of differentiability is called 1-differentiability on open interval (c, d).

2. The another type of differentiability known as 2-differentiability is defined as

$$\lim_{k \to 0^+} \frac{\tilde{\psi}(z_0 \ominus \tilde{\psi}(z_0 + k))}{-k} = \frac{\tilde{\psi}(z_0 - k) \ominus \tilde{\psi}(z_0)}{-k} = \tilde{\psi}'(z_0) \tag{9}$$

   Provided $\tilde{\psi}(z_0) \ominus \tilde{\psi}(z_0 + k)$ and $\tilde{\psi}(z_0 - k) \ominus \tilde{\psi}(z_0)$ must exist.

Definition 8: After the fuzzy differentiability, the another concept that we need in fuzzy integrability. Let us consider a function $\tilde{f}$ such that $\tilde{f} : [c, d] \longrightarrow R_F$. We say this function is Riemann integrable if given $\epsilon > 0\ \exists\ \delta > 0$ s.t $\forall\ Q = \{[\tilde{v}_1, \tilde{v}_2], \zeta\}$ of domain $[c, d]$

$$\Psi\Big(\sum_{Q}^{*} (\tilde{v}_1 - \tilde{v}_2) \odot \tilde{f}(\zeta) \Big) < \epsilon, \tag{10}$$

with norms $\Delta(P) < \delta$. The notation $\sum_P^*$ represents the addition operation $\oplus$ with respect to fuzzy numbers. In simple notation, if the function $\tilde{f}$ is Riemann-integrable than we write

$$I = (FR) \int_c^d \tilde{f}(z) dz. \tag{11}$$

2.1 Definition of fractional derivatives in fuzzy sense [17, 18, 19, 20, 21]

Here, the definitions of fractional derivative with power law kernel and Mittag-Leffler kernel are discussed in fuzzy sense. They are extended from the original definition by apply crisp definitions.

Fuzzy Caputo definition [22]: Choosing a function $h(\zeta) \in L_p^E \cap C^E$ where $L_p^E[c,d]$ contains all measurable function in fuzzy sense and $C^E[c,d]$ denotes the space of all continuous fuzzy valued functions. Then Caputo definition in g-differentiability fuzzy sense is given as

$$({}^{g}D_{a^+}^{\gamma} h)(\zeta) = \lim_{h \to 0} \frac{\psi(\zeta + k) \ominus_g \psi(\zeta)}{k}, \tag{12}$$

where

$$\psi(\zeta) = \frac{1}{\Gamma(1-\gamma)} \int_c^{\zeta} (\zeta - \eta)^{-\gamma} h(\eta) d\eta, < \gamma < 1. \tag{13}$$

In terms of p-level set, we can write as follows

$$({}^{g}D_{a^+}^{\gamma} \underline{h})(\zeta, p) = \frac{1}{\Gamma(1-\gamma)} \int_c^{\zeta} (\zeta - \eta)^{-\gamma} \underline{h}'(\eta, p) d\eta \tag{14}$$

$$({}^{g}D_{a^+}^{\gamma} \overline{h})(\zeta, p) = \frac{1}{\Gamma(1-\gamma)} \int_c^{\zeta} (\zeta - \eta)^{-\gamma} \overline{h}'(\eta, p) d\eta. \tag{15}$$

with

$$({}^{C}D_{a^+}^{\gamma} h)(\zeta, s) = \Big[({}^{g}D_{a^+}^{\gamma} \underline{h})(\zeta, p), ({}^{g}D_{a^+}^{\gamma} \overline{h})(\zeta, p) \Big]. \tag{16}$$

Fuzzy definition of Mittag-Leffler kernel fractional derivative:
The Atangana-Baleanu fractional derivative in fuzzy sense can be defined as follows

$$\left[{}^{ABC}D_{\zeta}^{\gamma} f(\zeta)\right]_s = \left[{}^{ABC}D_{\zeta}^{\gamma} \underline{f(\zeta, s)}, {}^{ABC}D_{\zeta}^{\gamma} \overline{f(\zeta, s)}\right], \tag{17}$$

where

$${}^{ABC}D_{\zeta}^{\gamma} f(\zeta) = \frac{B(\gamma)}{\Gamma(1-\gamma)} \int_0^{\zeta} E_{\gamma} \Big[\frac{-\gamma}{1-\gamma} (\zeta - \eta)^{\gamma} \Big] f'(\eta) d\eta. \tag{18}$$

2.2 Chebyshev polynomial and its basic properties

Here, we present the series definition of Chebyshev polynomial along with some of its properties. Chebyshev polynomial of degree i can be written as follows

$$\psi_i(\zeta) = i \sum_{k=0}^{i} \frac{(-1)^{i-k} (i+k-1)! 2^{2k}}{2k!(i-k)!} \zeta^k \tag{19}$$

where $i = 0, 1, \cdots$.
Chebyshev polynomials satisfies the orthogonality property on integral $[-1, 1]$ with respect to the weight function $\frac{1}{\sqrt{1-\zeta^2}}$ which can be represented as

$$\int_{-1}^{1} \frac{\psi_j(\zeta)\psi_i(\zeta)}{\sqrt{1-\zeta^2}} = \begin{cases} 0, & j \neq i, \\ \pi, & i = j = 0, \\ \frac{\pi}{2} & j = i \neq 0. \end{cases} \tag{20}$$

In this article, the considered interval is $[0, 1]$ thus the used Chebyshev polynomials are shifted on interval $[0, 1]$. According to the shift, the weight function transforms to $\frac{1}{\sqrt{\zeta-\zeta^2}}$.

2.3 Approximation of fuzzy valued function in terms of Chebyshev polynomial

Consider a fuzzy valued function $\tilde{u}(\zeta)$ which is a member of $L^E(0, 1)] \cap C^E(0, 1)$. It can be approximated in terms of shifted Chebyshev polynomials on the interval $[0, 1]$ [22]

$$\tilde{u}(\zeta) = \sum_{k=0}^{\infty} \tilde{d_k} \odot \varphi_k(\zeta), \tag{21}$$

the summation in above series is corresponds to the fuzzy addition $\oplus$. The unknown coefficients in this series can be determined by using the orthogonal property of Chebyshev polynomials

$$\tilde{d_k} = \frac{c_k}{\pi} \odot \int_0^1 \frac{\tilde{u}(\zeta) \odot \varphi_k(\zeta)}{\sqrt{\zeta - \zeta^2}} d\zeta \tag{22}$$

with c_k

$$c_k = \begin{cases} 1 & k = 0 \\ 2 & k = 1, 2, \cdots \end{cases} \tag{23}$$

We consider $(m + 1)$ terms of of series (39) for numerical computation, finite series can be rewritten in terms of matrix form as follows

$$\tilde{u}_m(\zeta) = \sum_{k=0}^{m} d_k \odot \varphi_k(\zeta) = \Theta_m^T \odot \Psi_m(\zeta), \tag{24}$$

where Θ_m^T denotes the unknown fuzzy coefficient vector and $\Psi_m(\zeta)$ is column vector having its elements shifted Chebyshev polynomials

$$\Theta_m^T = [d_0, d_1, \cdots, d_k] \tag{25}$$

$$\Psi_m(\zeta) = [\varphi_1(\zeta), \varphi_2(\zeta), \cdots, \varphi_m(\zeta)]. \tag{26}$$

3. Error bound

In this section, some lemmas and theorems have been proven. The error bound has been obtained with the help of lemmas.

Lemma 1: If $\tilde{h}(\zeta)$ is an absolute continuous function and ${}^{C}D^{k\gamma}_{a^+}\tilde{h}(\zeta)$ is a continuous function with $0 < \gamma < 1$, $k = 0, 1, \cdot, n+1$ then

$$[\tilde{h}(\zeta)]^r = [\underline{h^r}(\zeta), \overline{h^r}(\zeta)] \tag{27}$$

$$\underline{h^r}(\zeta) = \sum_{j=0}^{n} \frac{\zeta^{\gamma j}}{\Gamma(j\gamma+1)} {}^{C}D^{i\gamma}\underline{h^r}(0^+) + \frac{{}^{C}D^{(n+1)\gamma}h^{\gamma}(\zeta_0)}{\Gamma(n\gamma+\gamma+1)}\zeta^{(n+1)\gamma}, \tag{28}$$

$$\overline{h^r}(\zeta) = \sum_{j=0}^{n} \frac{\zeta^{\gamma j}}{\Gamma(j\gamma+1)} {}^{C}D^{i\gamma}\overline{h^r}(0^+) + \frac{{}^{C}D^{(n+1)\gamma}h^{\gamma}(\zeta_0)}{\Gamma(n\gamma+\gamma+1)}\zeta^{(n+1)\gamma}, \tag{29}$$

with ${}^{C}D^{i\gamma}\overline{h^r}(0^+) = {}^{C}D^{i\gamma}\overline{h^r}(\zeta)|_{\zeta=0}$ and ${}^{C}D^{i\gamma}\underline{h^r}(0^+) = {}^{C}D^{i\gamma}\underline{h^r}(\zeta)|_{\zeta=0}$.

Proof: For proof see article [23].

Lemma 2: Considering a function $\tilde{u}$ which belongs to both the spaces $(\zeta) \in L^E(0,1]$ $C^E(0,1]$ and its fractional derivative in Caputo sense ${}^{C}D^{\gamma}\tilde{u}(\zeta)$ is a member of space $C^E(0,1]$ with $\gamma \in (0,1)$, $\zeta \in [0,1]$. If the function $\tilde{u}(\zeta)$ satisfies the conditions of 1-differentiability then

$$|\underline{\tilde{u}^r}(\zeta) - \sum_{j=0}^{n} \frac{\zeta^{\gamma j}}{\Gamma(j\gamma+1)} {}^{C}D^{i\gamma}\underline{\tilde{u}^r}(0^+)| \leq \underline{M}^r_{\gamma}\frac{\zeta^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}, \tag{30}$$

$$|\overline{\tilde{u}^r}(\zeta) - \sum_{j=0}^{n} \frac{\zeta^{\gamma j}}{\Gamma(j\gamma+1)} {}^{C}D^{i\gamma}\overline{\tilde{u}^r}(0^+)| \leq \overline{M}^r_{\gamma}\frac{\zeta^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}, \tag{31}$$

where $D^{(n+1)\gamma}\overline{\tilde{u}}(\zeta_0) < \overline{M}_{\gamma}$ and $D^{(n+1)\gamma}\underline{\tilde{u}}(\zeta_0) < \underline{M}_{\gamma}$.

Proof: The following expression can be obtained with the help of Lemma 1

$$\underline{u^r}(\zeta) = \sum_{j=0}^{n} \frac{\zeta^{\gamma j}}{\Gamma(j\gamma+1)} {}^{C}D^{i\gamma}\underline{u^r}(0^+) + \frac{{}^{C}D^{(n+1)\gamma}u^{\gamma}(\zeta_0)}{\Gamma(n\gamma+\gamma+1)}\zeta^{(n+1)\gamma}, \tag{32}$$

$$\overline{u^r}(\zeta) = \sum_{j=0}^{n} \frac{\zeta^{\gamma j}}{\Gamma(j\gamma+1)} {}^{C}D^{i\gamma}\overline{u^r}(0^+) + \frac{{}^{C}D^{(n+1)\gamma}u^{\gamma}(\zeta_0)}{\Gamma(n\gamma+\gamma+1)}\zeta^{(n+1)\gamma}. \tag{33}$$

By using the properties of mod function

$$|\underline{u^r}(\zeta) - \sum_{j=0}^{n} \frac{\zeta^{\gamma j}}{\Gamma(j\gamma+1)} {}^{C}D^{i\gamma}\underline{u^r}(0^+)| \leq \frac{|{}^{C}D^{(n+1)\gamma}u^{\gamma}(\zeta_0)|}{\Gamma(n\gamma+\gamma+1)}\zeta^{(n+1)\gamma} \tag{34}$$

$$\leq \overline{M}_{\gamma}\frac{\zeta^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)} \tag{35}$$

The expression for upper bound can derived in similar manner.

Theorem 1: Considering the fractional derivative of order γ of function $\tilde{u}(\zeta)$ belonging to both of spaces $L^E(0,1]$ and $C^E(0,1]$ and space of fuzzy numbers with finite dimension. If $\phi_0(\zeta), \phi_1(\zeta), \cdots, \phi_n(\zeta)$ is the basis of $L^E(0,1] \cap C^E(0,1]$. Le $\tilde{u}_n(\zeta) = \Theta_n^T \odot \Psi_n(\zeta)$ is the best approximation of function $\tilde{u}(\zeta)$ in terms of Chebyshev polynomials then

$$\lim_{n\to\infty} D(\tilde{u}(\zeta), \tilde{u}_n(\zeta) = 0 \tag{36}$$

Proof: Let $h(\zeta)$ is a fuzzy valued function with its r-level set $[\tilde{h}(\zeta)]^r = [\underline{h^r}(\zeta), \overline{h^r}(\zeta)]$. By the Taylor's formula given in Lemma 1 we have

$$|\underline{\tilde{u}^r}(\zeta) - \underline{\tilde{h}^r}(\zeta| \leq \underline{M}^r_\gamma \frac{\zeta^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}, \tag{37}$$

$$|\overline{\tilde{u}^r}(\zeta) - \overline{\tilde{h}^r}(\zeta)| \leq \overline{M}^r_\gamma \frac{\zeta^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}. \tag{38}$$

The function $\tilde{u}(\zeta)$ has its best approximation$\Theta_n^T \odot \Psi_n(\zeta)$ then we have

$$|\underline{\tilde{u}^r}(\zeta) - \underline{\Theta}_n^T \odot \Psi_n(\zeta)| \leq |\underline{\tilde{u}^r}(\zeta) - \underline{\tilde{h}^r}(\zeta| \leq \underline{M}^r_\gamma \frac{\zeta^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}, \tag{39}$$

$$|\overline{\tilde{u}^r}(\zeta) - \overline{\Theta}_n^T \odot \Psi_n(\zeta)| \leq |\overline{\tilde{u}^r}(\zeta) - \overline{\tilde{h}^r}(\zeta)| \leq \overline{M}^r_\gamma \frac{\zeta^{(n+1)\gamma}}{\Gamma(n\gamma+\gamma+1)}. \tag{40}$$

Now

$$\begin{aligned} \|\underline{\tilde{u}^r}(\zeta) - \underline{\Theta}_n^T \odot \Psi_n(\zeta)\|^2 &\leq \|\underline{\tilde{u}^r}(\zeta) - \underline{\tilde{h}^r}(\zeta\|^2 \\ &\leq \frac{\underline{M}^{r\,2}_\gamma}{(\Gamma(n\gamma+\gamma+1))^2} \int_0^1 \zeta^{2(n+1)\gamma} d\zeta, \\ &\leq \frac{\underline{M}^{r\,2}_\gamma}{(\Gamma(n\gamma+\gamma+1))^2 (2(n+1)\gamma+1)^2}. \end{aligned} \tag{41}$$

In this similar way

$$\begin{aligned} \|\overline{\tilde{u}^r}(\zeta) - \overline{\Theta}_n^T \odot \Psi_n(\zeta)\|^2 &\leq \|\overline{\tilde{u}^r}(\zeta) - \overline{\tilde{h}^r}(\zeta\|^2 \\ &\leq \frac{\overline{M}^{r\,2}_\gamma}{(\Gamma(n\gamma+\gamma+1))^2} \int_0^1 \zeta^{2(n+1)\gamma} d\zeta, \\ &\leq \frac{\overline{M}^{r\,2}_\gamma}{(\Gamma(n\gamma+\gamma+1))^2 (2(n+1)\gamma+1)^2}. \end{aligned} \tag{42}$$

The right hand side term of equation (35) and (36) tends to zero as n tends to ∞. Thus we can say that

$$\lim_{n\to\infty} D(\tilde{u}(\zeta), \tilde{u}_n(\zeta) = 0$$

4. Model description and numerical method dealing with model

In this section, we have investigated a fuzzy model of reaction-diffusion equation. This model has a time delay and time fractional derivative of power law kernel and Mittag-Leffler kernel type.

$$ {}_{0}^{GF_i}D_t^{\beta}\tilde{u}(\zeta,\eta) = \tilde{D}\frac{\partial^2\tilde{u}(\zeta,\eta)}{\partial\zeta^2} \oplus d_1\kappa(u(\tilde{\zeta},\eta), u(\zeta,\tilde{\eta}-\epsilon)) \oplus \tilde{f}(\zeta,\eta), \tag{43} $$

with $0 < \zeta < 1$, $0 < \eta < 1$ and $\epsilon \in [-\lambda, 0]$. We investigated this model with two type of fractional derivative

- $GF_1{=}C$, Caputo derivative with power law kernel.
- $GF_2{=}ABC$, Caputo derivative with Mittag- Leffler kernel.

For model, the following initial and boundary conditions are considered

$$ \tilde{u}(\zeta,0) = \tilde{f}_1(\zeta), \tilde{u}(0,\eta) = \tilde{f}_2(\eta), \tilde{u}(1,\eta) = \tilde{f}_3(\eta). \tag{44} $$

Here all fractional derivatives are considered in respect of Hukuhara derivative. The model can be fuzzified $\forall\ s \in [0,1]$ in the following manner

$$ [\tilde{u}(\zeta,\eta)]^s = [\underline{u}(\zeta,\eta,s), \overline{u}(\zeta,\eta,s)], \tag{45} $$
$$ [{}_{0}^{GF_i}D_t^{\beta}\tilde{u}(\zeta,\eta)]^s = [{}_{0}^{GF_i}D_t^{\beta}\underline{u}(\zeta,\eta), {}_{0}^{GF_i}D_t^{\beta}\overline{u}(\zeta,\eta)], \tag{46} $$
$$ [\frac{\partial^2\tilde{u}(\zeta,\eta)}{\partial\zeta^2}]^s = [\frac{\partial^2\underline{u}(\zeta,\eta,s)}{\partial\zeta^2}, \frac{\partial^2\overline{u}(\zeta,\eta,s)}{\partial\zeta^2}], \tag{47} $$
$$ [\tilde{f}(\zeta,\eta)]^s = [\underline{f}(\zeta,\eta,s), \overline{u}(\zeta,\eta,s)]. \tag{48} $$

The model can be written in terms of lower and bound as

$$ {}_{0}^{GF_i}D_t^{\beta}\underline{u}(\zeta,\eta) = \tilde{D}\frac{\partial^2\underline{u}(\zeta,\eta,s)}{\partial\zeta^2} \oplus d_1\kappa(\underline{u}(\zeta,\eta), \underline{u}(\zeta,\eta-\epsilon)) \oplus \underline{f}(\zeta,\eta,s), \tag{49} $$
$$ \underline{u}(\zeta,0) = \underline{f}_1(\zeta), \tag{50} $$
$$ \underline{u}(0,\eta) = \underline{f}_2(\eta), \tag{51} $$
$$ \underline{u}(1,\eta) = \underline{f}_3(\eta), \tag{52} $$

$$ {}_{0}^{GF_i}D_t^{\beta}\overline{u}(\zeta,\eta) = \tilde{D}\frac{\partial^2\overline{u}(\zeta,\eta,s)}{\partial\zeta^2} \oplus d_1\kappa(\overline{u}(\zeta,\eta), \overline{u}(\zeta,\eta-\epsilon)) \oplus \overline{f}(\zeta,\eta,s), \tag{53} $$
$$ \overline{u}(\zeta,0) = \overline{f}_1(\zeta), \tag{54} $$
$$ \overline{u}(0,\eta) = \overline{f}_2(\eta), \tag{55} $$
$$ \overline{u}(1,\eta) = \overline{f}_3(\eta). \tag{56} $$

To find out the numerical solution of model 43, we use the Chebyshev approximation of function $\tilde{u}(\zeta,\eta)$

$$ \tilde{u}(\zeta,\eta) = \tilde{u}_m(\zeta,\eta) = \sum_{j=0}^{m}\sum_{k=0}^{m} \tilde{c}_{jk} \odot \phi_j(\zeta) \odot \phi_k(\eta) = \Psi_m^{\eta}(\zeta) \odot \tilde{\Theta}_m \odot \Psi_m(\zeta), \tag{57} $$

where $\tilde{c}_{jk}$ are to be determined. The model 43 has the fractional order derivatives of order β and γ. Thus approximating these fractional derivatives in terms of Chebyshev series. For this we operate fractional differential operator of β and γ with power kernel law on both side of equation 57

$$\begin{aligned}
{}_0^C D_t^\beta \tilde{u}(\zeta,\eta) &= {}_0^C D_t^\beta \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_j(\zeta) \odot \phi_p(\eta),\\
&= \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_j(\zeta) \odot {}_0^C D_t^\beta \phi_p(\eta),\\
&= \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_j(\zeta) \odot {}_0^C D_t^\beta \sum_{k=0}^{p} \frac{p(-1)^{p-k}(p+k-1)!2^{2k}}{2k!(p-k)!}\eta^k,\\
&= \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_j(\zeta) \odot \sum_{k=0}^{p} \frac{p(-1)^{p-k}(p+k-1)!2^{2k}}{2k!(p-k)!} \odot \frac{\Gamma(1+k)}{\Gamma(1+k-\beta)}\eta^{k-\beta},\\
&= \sum_{j=0}^{m}\sum_{p=0}^{m}\sum_{k=0}^{p} \tilde{c}_{jp} \odot \phi_j(\zeta) \odot \frac{p(-1)^{p-k}(p+k-1)!2^{2k}}{2k!(p-k)!} \odot \frac{\Gamma(1+k)}{\Gamma(1+k-\beta)}\eta^{k-\beta}.
\end{aligned} \tag{58}$$

Applying same procedure for the fractional derivative with Mittag-Leffler kernel

$$\begin{aligned}
{}_0^{ABC} D_t^\beta \tilde{u}(\zeta,\eta) &= {}_0^C D_t^\beta \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_j(\zeta) \odot \phi_p(\eta),\\
&= \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_j(\zeta) \odot {}_0^{ABC} D_t^\beta \phi_p(\eta),\\
&= \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_j(\zeta) \odot {}_0^{ABC} D_t^\beta \sum_{k=0}^{p} \frac{p(-1)^{p-k}(p+k-1)!2^{2k}}{2k!(p-k)!}\eta^k,\\
&= \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_j(\zeta) \odot \sum_{k=0}^{p} \frac{p(-1)^{p-k}(p+k-1)!2^{2k}}{2k!(p-k)!} \odot \varpi(k,\beta,\eta).
\end{aligned} \tag{59}$$

where the ABC derivative of function η^k can be evaluated by using any numerical integration scheme and we have taken ${}_0^{ABC} D_t^\beta \eta^k = \varpi(k,\beta,\eta)$ Now we approximate the spatial derivative and delay function with the help of Chebyshev polynomial series expansion

$$\begin{aligned}
\frac{\partial^2 \tilde{u}(\zeta,\eta)}{\partial \zeta^2} &= {}_0^C D_\zeta^2 \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_j(\zeta) \odot \phi_p(\eta),\\
&= \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_p(\eta) \odot {}_0^C D_\zeta^2 \phi_j(\zeta),\\
&= \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_p(\eta) \odot {}_0^C D_\zeta^2 \sum_{k=0}^{j} \frac{j(-1)^{j-k}(j+k-1)!2^{2k}}{2k!(j-k)!}\zeta^k,\\
&= \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_p(\eta) \odot \sum_{k=0}^{j} \frac{j(-1)^{j-k}(j+k-1)!2^{2k}}{2k!(j-k)!}k(k-1)\zeta^{k-2}.
\end{aligned} \tag{60}$$

And

$$\tilde{u}(\zeta,\eta-\epsilon) = \sum_{j=0}^{m}\sum_{p=0}^{m} \tilde{c}_{jp} \odot \phi_j(\zeta) \odot \sum_{k=0}^{p} \frac{p(-1)^{p-k}(p+k-1)!2^{2k}}{2k!(p-k)!}(\eta-\epsilon)^k \tag{61}$$

The fuzzy initial and boundary conditions are approximated with the help of equation 44

$$\sum_{j=0}^{m}\sum_{p=0}^{m}\tilde{c}_{jp}\odot\phi_j(\zeta)\odot\phi_p(0)=\tilde{f}_1(\zeta), \tag{62}$$

$$\sum_{j=0}^{m}\sum_{p=0}^{m}\tilde{c}_{jp}\odot\phi_j(0)\odot\phi_p(\eta)=\tilde{f}_2(\eta), \tag{63}$$

$$\sum_{j=0}^{m}\sum_{p=0}^{m}\tilde{c}_{jp}\odot\phi_j(1)\odot\phi_p(\eta)=\tilde{f}_3(\eta). \tag{64}$$

The following residue is obtained for the case of power law kernel fractional derivative after putting the value of unknown function and its derivatives in model

$$\begin{aligned}\tilde{\xi}_c(\zeta,\eta)=&\sum_{j=0}^{m}\sum_{p=0}^{m}\sum_{k=0}^{p}\tilde{c}_{jp}\odot\phi_j(\zeta)\odot\frac{p(-1)^{p-k}(p+k-1)!2^{2k}}{2k!(p-k)!}\odot\frac{\Gamma(1+k)}{\Gamma(1+k-\beta)}\eta^{k-\beta}\\&-\tilde{D}\sum_{k=0}^{j}\frac{j(-1)^{j-k}(j+k-1)!2^{2k}}{2k!(j-k)!}k(k-1)\zeta^{k-2}\\&\oplus d_1\kappa\left(\sum_{j=0}^{m}\sum_{k=0}^{m}\tilde{c}_{jk}\odot\phi_j(\zeta)\odot\phi_k(\eta),\sum_{j=0}^{m}\sum_{p=0}^{m}\tilde{c}_{jp}\odot\phi_j(\zeta)\odot\sum_{k=0}^{p}\frac{p(-1)^{p-k}(p+k-1)!2^{2k}}{2k!(p-k)!}(\eta-\epsilon)^k\right)\\&\oplus\tilde{f}(\zeta,\eta)\end{aligned} \tag{65}$$

Similarly, for Mittag-Leffler kernel fractional derivative the following residue is obtained

$$\begin{aligned}\tilde{\xi}_{ABC}(\zeta,\eta)=&\sum_{j=0}^{m}\sum_{p=0}^{m}\sum_{k=0}^{p}\tilde{c}_{jp}\odot\phi_j(\zeta)\odot\frac{p(-1)^{p-k}(p+k-1)!2^{2k}}{2k!(p-k)!}\odot\varpi(k,\beta,\eta)\\&-\tilde{D}\sum_{k=0}^{j}\frac{j(-1)^{j-k}(j+k-1)!2^{2k}}{2k!(j-k)!}k(k-1)\zeta^{k-2}\\&\oplus d_1\kappa\left(\sum_{j=0}^{m}\sum_{k=0}^{m}\tilde{c}_{jk}\odot\phi_j(\zeta)\odot\phi_k(\eta),\sum_{j=0}^{m}\sum_{p=0}^{m}\tilde{c}_{jp}\odot\phi_j(\zeta)\odot\sum_{k=0}^{p}\frac{p(-1)^{p-k}(p+k-1)!2^{2k}}{2k!(p-k)!}(\eta-\epsilon)^k\right)\\&\oplus\tilde{f}(\zeta,\eta).\end{aligned} \tag{66}$$

We collocate equations $62-64$, 65 and $62-64$, 66 at Chbeyshev nodes respectively for power law kernel and Mittag-leffler kernel fractional derivatives. A system of fuzzy algebraic equation is obtained which gives the value of unknowns after solving by the method given in [24].

5. Numerical examples

In this section, the proposed method has been applied to some particular cases of our fuzzy reaction-diffusion model. We have used the Mathematica 11.3 in all numerical calculations.

Example 1: Choosing $\beta=0.9$, $\tilde{D}=1$, $d_1=1$ $\kappa(u(\tilde{\zeta,\eta}),u(\zeta,\tilde{\eta-\epsilon}))=-\tilde{u}^2+\tilde{u}(\zeta,\eta-0.4)$ and

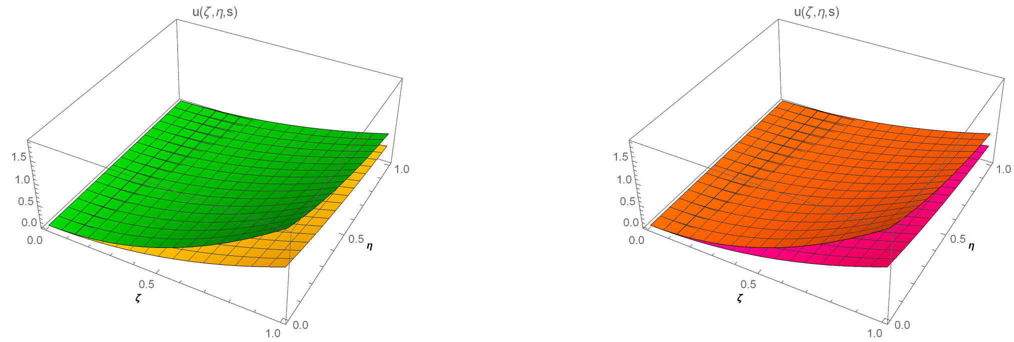
$\epsilon = 0.4$, the following fuzzy time fractional diffusion equation is obtained

$$ {}_0^C D_t^{0.9}\tilde{u}(\zeta,\eta) = \frac{\partial^2 \tilde{u}(\zeta,\eta)}{\partial \zeta^2} \ominus \tilde{u}^2 + \tilde{u}(\zeta,\eta-0.4) \oplus \tilde{f}(\zeta,\eta). \tag{67} $$

We choose the following initial and boundary conditions

$$ \tilde{u}(\zeta,0) = \tilde{\alpha}(s)\zeta^2, \tilde{u}(0,\eta) = 0, \tilde{u}(1,\eta) = \tilde{\alpha}(s)e^{-\eta} \tag{68} $$

with exact solution $\tilde{u}(\zeta,\eta,\beta,s) = \tilde{\alpha}(s)\zeta^2 e^{-\eta}$ and $\tilde{\alpha}(s) = [1+0.5s, 2-0.5s]$. We plot the graph of numerical and exact solution for $m = 6$ which can be seen in Fig.1. Table 1 depicts the variation of absolute error for lower and upper solution both.



(a) (b)

Figure 1: Plots of $u(\zeta,\eta,s)$ for $m = 4$ in case of numerical and exact solution for $s = 0.2$.

Table 1: Exhibition of absolute error at different value of ζ at $\eta = 0.1$, $s = 0.1$ and for $m = 7$.

$\zeta \downarrow$	Value of absolute error in case of lower solution	Value of absolute error in case of upper solution
0.1	6.6×10^{-5}	4.5×10^{-5}
0.2	4.8×10^{-5}	6.4×10^{-5}
0.3	5.4×10^{-5}	7.2×10^{-5}
0.4	1.4×10^{-5}	3.0×10^{-5}
0.5	4.9×10^{-5}	6.9×10^{-5}
0.6	7.4×10^{-5}	2.2×10^{-5}
0.7	2.5×10^{-5}	4.5×10^{-5}
0.8	9.1×10^{-6}	8.7×10^{-6}

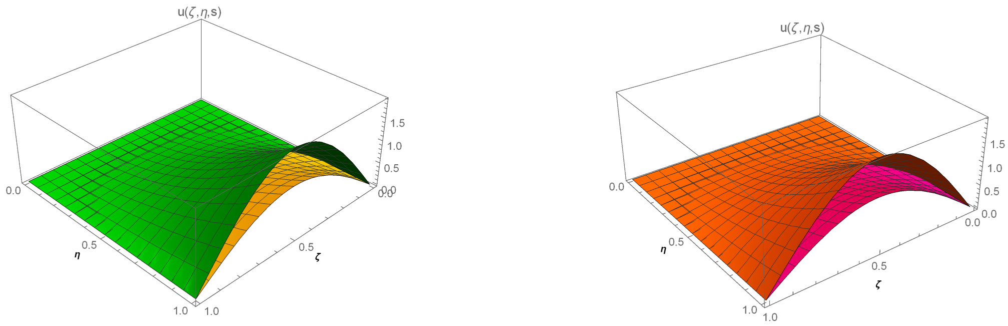
Example 2: Considering another particular case of diffusion equation with $\beta = 0.9$, $\tilde{D} = 1$, $d_1 = 1$, $\kappa(u(\tilde{\zeta},\eta), u(\zeta,\widetilde{\eta-\epsilon})) = \tilde{u}(\zeta,\eta)\big(1-\tilde{u}(\zeta,\eta-0.4)\big)$ and $\epsilon = 0.4$

$$ {}_0^C D_t^{0.9}\tilde{u}(\zeta,\eta) = \frac{\partial^2 \tilde{u}(\zeta,\eta)}{\partial \zeta^2} \oplus \tilde{u}(\zeta,\eta)\big(1-\tilde{u}(\zeta,\eta-0.4)\big) \oplus \tilde{f}(\zeta,\eta). \tag{69} $$

The fuzzy initial and boundary conditions are taken as follows

$$\tilde{u}(\zeta,0)=0,\tilde{u}(0,\eta)=0,\tilde{u}(1,\eta)=0. \tag{70}$$

The exact solution od above problem is $\tilde{u}(\zeta,\eta,s)=\tilde{\alpha}(s)\eta^2\sin(\pi\zeta)$ and the value of function $\tilde{\alpha}(s)$ is $[1+0.5s, 2-0.5s]$. Fig.2 represents the 3D plot of exact and numerical solution. The absolute error between exact and obtained numerical solution can be seen in Table 2.



(a) (b)

Figure 2: Plots of $u(\zeta,\eta,s)$ for $m=4$ in case of numerical and exact solution for $s=0.2$.

Table 2: Exhibition of absolute error at different value of ζ at $\eta=0.1$, $s=0.1$ and for $m=7$.

$\zeta\downarrow$	Value of absolute error in case of lower solution	Value of absolute error in case of upper solution
0.1	3.5×10^{-4}	1.6×10^{-4}
0.2	2.8×10^{-4}	6.3×10^{-4}
0.3	7.5×10^{-3}	9.4×10^{-4}
0.4	1.7×10^{-3}	7.3×10^{-3}
0.5	6.7×10^{-3}	5.6×10^{-3}
0.6	5.9×10^{-3}	2.8×10^{-3}
0.7	3.3×10^{-4}	8.6×10^{-4}
0.8	7.2×10^{-4}	4.9×10^{-4}

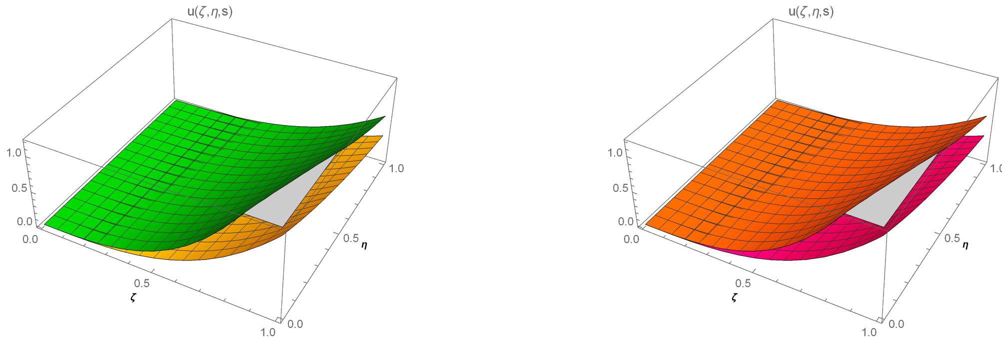
Example 3: If we choose $\beta=0.9$, $\tilde{D}=1$, $d_1=1$, $\kappa(u(\tilde{\zeta,\eta}),u(\zeta,\tilde{\eta-\epsilon}))=\tilde{u}(\zeta,\eta)\big(1-\tilde{u}(\zeta,\eta-0.4)\big)$ and $\epsilon=0.4$ with Mittag-Leflfer time fractional derivative then the following fuzzy reaction-diffusion equation is obtained

$${}_0^{ABC}D_t^{0.9}\tilde{u}(\zeta,\eta)=\frac{\partial^2\tilde{u}(\zeta,\eta)}{\partial\zeta^2}\oplus\tilde{u}(\zeta,\eta)\big(1-\tilde{u}(\zeta,\eta-0.4)\big)\oplus\tilde{f}(\zeta,\eta). \tag{71}$$

The following initial and boundary conditions is chosen

$$\tilde{u}(\zeta,0)=\tilde{\alpha}(s)\zeta^3,\tilde{u}(0,\eta)=0,\tilde{u}(1,\eta)=\tilde{\alpha}(s)e^{-\eta}. \tag{72}$$

where $\tilde{\alpha}(s)$ is $[1+0.5s, 2-0.5s]$. The exact solution of this problem is $\tilde{\alpha}(s)e^{-\eta}\zeta^3$. Fig.3 is corresponds to the numerical and exact solution of this problem. Table 3 represents the absolute error between numerical and exact solution in case of lower and upper solution both.



(a) (b)

Figure 3: Plots of $u(\zeta, \eta, s)$ for $m = 4$ in case of numerical and exact solution for $\eta = 0.5$.

Table 3: Exhibition of absolute error at different value of ζ at $\eta = 0.1$, $s = 0.1$ and for $m = 7$.

$\zeta \downarrow$	Value of absolute error in case of lower solution	Value of absolute error in case of upper solution
0.1	2.4×10^{-3}	6.4×10^{-3}
0.2	3.1×10^{-3}	2.4×10^{-3}
0.3	2.3×10^{-4}	1.6×10^{-3}
0.4	8.5×10^{-4}	7.5×10^{-4}
0.5	2.4×10^{-3}	5.9×10^{-3}
0.6	4.5×10^{-3}	6.5×10^{-3}
0.7	5.5×10^{-3}	1.4×10^{-3}
0.8	7.9×10^{-4}	1.3×10^{-3}

We can draw a conclusion from all 3 examples that our method has successfully implemented on a new class of fuzzy time delay partial differential equation with power law and Mittag-Leffler kernel fractional derivative. We see that our method is valid for both type of delay differential equation having Caputo and ABC fractional derivative.

Example 4**:** Considering the following model from the research paper

$$\frac{\partial^\alpha \tilde{u}(\zeta, \eta)}{\partial \eta^\alpha} = \frac{\partial^2 \tilde{u}(\zeta, \eta)}{\partial \zeta^2}, \quad 0 < \zeta < 1, \, \eta > 0 \tag{73}$$

subject to the boundary conditions $\tilde{u}(0, \eta) = 0$, $\tilde{u}(1, \eta) = 0$, and following initial condition

$$\tilde{u}(\zeta, 0) = \Big\{ k \sin(\pi \zeta), \quad 0 \le \zeta \le 1, \tag{74}$$

where $\tilde{k}(r) = (0.1r - 0.1, 0.1 - 0.1r) \quad \forall r \in [0, 1]$ and exact solution for above problem is as follows:

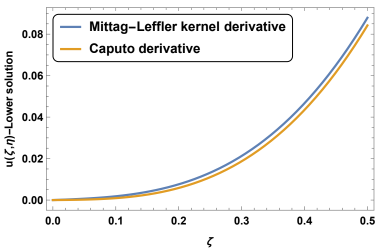
$$\tilde{u}(\zeta, \eta; \alpha, r) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} \eta^{n\alpha}}{\Gamma(1 + n\alpha)} \tilde{k}(r) \sin(\pi\zeta) \tag{75}$$

The comaparision between error obtanied from mentioned liteartture and our results is depicted with the following absolute error table. From the analysis presented in the table above,

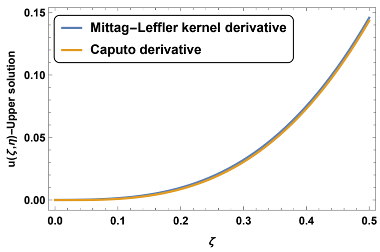
r	Error in article	Error obtained from our method
0	0.0005933599789698339	2.65 $\times 10^{-5}$
0.2	0.0004746879831758668	3.54 $\times 10^{-6}$
0.4	0.0003560159873818988	7.14 $\times 10^{-6}$
0.6	0.00023734399158793383	5.47 $\times 10^{-6}$
0.8	0.00011867199579396691	6.14 $\times 10^{-6}$

Table 4: Values of absolute error for lower solution for different r-levels at $\eta = 0.05$, $\alpha = 0.5$, and for all $r \in [0, 1]$.

it becomes evident that our method exhibits superior accuracy, particularly in the context of Caputo type fractional derivative models, as considered in existing literature. Furthermore, our approach consistently delivers the desired levels of precision when applied to complex Mittag-Leffler kernel fractional fuzzy models, which are the focus of our investigation. These findings underscore the effectiveness and applicability of our method across a range of fractional calculus models, highlighting its potential for various real-world applications.

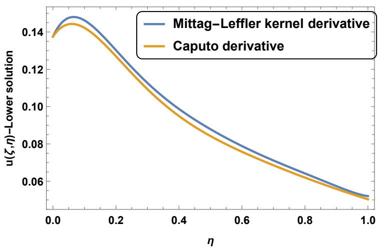


(a) Lower Solution

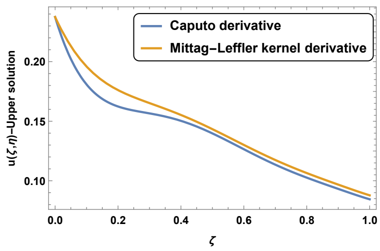


(b) Upper Solution

Figure 4: Comparison of solutions of $u(\zeta, \eta, s)$ for $m = 4$ in case of Mittag-Leffler kernel derivative and Caputo derivative for $\eta = 0.5$ and $r = 0.2$.



(a) Lower Solution



(b) Upper Solution

Figure 5: Plots of $u(\zeta, \eta, s)$ for $m = 4$ in case of numerical and exact solution for $\zeta = 0.5$ and $r = 0.2$.

A comprehensive comparative investigation has been conducted on reaction-diffusion models incorporating time-delay, juxtaposing the Caputo and Mittag-Leffler kernel derivatives. At a fixed time parameter of $t = 0.5$, the concentration of solute, represented by $u(\zeta, \eta, s)$, exhibits notably higher values when employing the Mittag-Leffler kernel derivative, evident across both upper and lower solution boundaries. However, this discrepancy in concentration diminishes when considering the upper solution. Similarly, when observing a fixed spatial position, the concentration of solute is consistently higher when employing the Mittag-Leffler kernel compared to the Caputo case. Notably, the introduction of time delay induces a pronounced bending in the plot of $u(\zeta, \eta, s)$, particularly evident until the time delay of $t = 0.4$ has transpired.

6. Conclusion

In this article, a study of fractional reaction-diffusion equation has been done with time delay in fuzzy environment. We have taken time-fractional derivative with power law kernel and Mittag-Leffler kernel. The approximation of unknown fuzzy function is based upon Chebyshev polynomials. The spectral method has been used in solving the model. This method converted the model into a system of fuzzy algebraic equations. That is solved with the help of method given in literature. The feasibility of proposed method can be seen from the exact and numerical solution figures. We can say our method is applicable on such type of models and has a desired accuracy.

7. References

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