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Approximate solution of a class of two-dimensional non-linear variable order fractional reaction-diffusion equations in porous media

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Abstract

In the present scientific article, an efficient operational matrix based on the famous Bernstein polynomials is applied for the numerical solution of two-dimensional non-linear variable order reaction-diffusion equation in porous media with given initial and boundary conditions. An operational matrix is constructed for fractional variable order differentiation w.r.to space variable x, y and time t, so that our proposed model is converted into a system of non-linear algebraic equations with the help of collocation method, which can be solved employing the Newton-Iteration method. The salient features of the article are finding the stability analysis and error bounds of the proposed method and also the validation and the effectiveness of the method through the RMS, L_∞ and L_2 errors. The physical presentation of the these errors for considered two-dimensional non-linear variable order reaction-diffusion with their exact solutions shows the method is too good for finding the solution of these kind of problems.

1. Introduction

The variable order differentiation has a vast scope in applied mathematics and allied disciplines, it is an extension of fractional order differentiation which is an extension of integer order differentiation. Now a days the scope of variable order partial differential equations (PDEs) is widely found in various aspects of mathematics and science. An wide explanation of fractional calculus can be studied from many literature, viz., [2, 22, 40, 41, 44]. Young researchers, engineers, scientists and mathematicians are found a wide application of fractional calculus theory in numerical biology, medical science, chemistry, image processing and mathematical physics etc.

The integer order differentiation are being used for short term memory systems, the fractional order derivative for constant order are being used for long term systems. Moreover we use the variable order derivative to characterize the variable memory systems. Thus the variable order derivative has many application to model several physical and biological phenomenon more accurately.

The wide application of fractional derivatives drag the focus of mathematician to find the exact and numerical solutions of the mathematical model of many non-linear physical and biological process. As the exact solution of many fractional partial differential equations do not exist or very tough to find their exact solution so there are many numerical scheme has been developed to find their approximate numerical solutions [17,23,24,39]. Many numerical scheme have been developed to find the approximate numerical solution of fractional order PDEs [13, 16, 36, 38]. Few of very commonly used numerical schemes are based on differential transform technique [1], Adomain decomposition scheme [10], homotopy perturbation method [19] etc. Some researchers have been developed the operational matrix technique to find the numerical solution of fractional PDEs. These techniques basically based on the sine wavelets [20], Chebyshev wavelets [3], Haar wavelets [26], Legendre wavelets [5] etc. Few operational matrix are developed on polynomials such as Genocchi polynomial [25], Chebyshev polynomial, Luca polynomial, Fibonacci polynomials etc. In the mathematical modelling of many complex models we use non linear PDEs of variable order.

This new concept of fractional calculus theory can be used in many branches of science, signal and control processing, mechanics etc [35,42,43]. Due to the presence of variable order fractional operator brings the complex kernels for variable powers do it is hard to calculate the solution by numerical techniques rather than constant order fractional derivatives. Chen et al. [12], presented a new approach of collocation method based on boundary type radial basis function to find the the numerical solution of fractional order diffusion equation. In the article [11], a collocation method based on domain type radial basis function has been applied to a constant and variable order derivative to find an approximate solution. To find the numerical solution of variable order fractional differential equation a finite difference scheme has been proposed in [27] with stability and convergence analysis of scheme. Moreover there are many numerical scheme have been proposed to find the numerical solution of variable order fractional differentiation viz., B-linear spline technique [30], discretization technique [15], integro quadratic spline interpolations techniques [21], cubic spline technique [32], finite difference method [33], spectral

collocation technique [46] etc.
A significant partition of fractional calculus theory is the two-dimensional fractional differential equations and integral equations which have been further studied in the present article. In [8], author have been solved the two-dimensional fractional percolation equation in non homogeneous porous media. In the present article, we have proposed an accurate and efficient numerical technique to solve two dimensional non-linear variable order reaction-diffusion equation arising in porous media. The two dimensional flow of the solute in the porous media are governed under the hypothesis of continuity and Darcy's law is given by Bear and Verruijt [4].
In order to analyses the diffusion of the solute in porous media in two-dimension, we will try to develop a [7] two-dimensional sub-diffusion physical model in the present article where the variable order derivative is taken in Caputo sense as

$$\begin{aligned}\frac{\partial^{\mu_1(x,y,t)}\zeta(x,y,t)}{\partial t^{\mu_1(x,y,t)}} &= p_1(x,y,t)\frac{\partial^{\mu_2(x,y,t)}\zeta(x,y,t)}{\partial x^{\mu_2(x,y,t)}} + p_2(x,y,t)\triangle\zeta(x,y,t) \\ &\quad - p_3(x,y,t)\triangledown\zeta(x,y,t) + p_4(\zeta(x,y,t)) + p_5(x,y,t),\end{aligned} \tag{1}$$

with the following initial and boundary conditions:

$$\begin{aligned}\zeta(x,y,0) =&\lambda_1(x,y), \zeta(0,y,t) = \lambda_2(y,t), \ \ \zeta_x(1,y,t) = \lambda_3(y,t), \\ &\zeta_y(x,0,t) = \lambda_4(x,t) \text{ and } \zeta_y(x,1,t) = \lambda_5(x,t).\end{aligned} \tag{2}$$

In this concerned mathematical model (1)-(2), $\mu_1(x,y,t)$ and $\mu_2(x,y,t)$ denotes the order of derivative which depends on space variable and time variable such that $g-1 < \mu_1(x,y,t), \mu_2(x,y,t) < g$. Here g is the first integer not less than $\mu_1(x,y,t)$ and $\mu_2(x,y,t)$. The symbol $\triangle$ and $\triangledown$ denotes the Laplacian operator $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ and gradient operator $(\frac{\partial}{\partial x} \ \ \frac{\partial}{\partial y})$ respectively. The other terms $p_{i's}(x,y,t)$ and $\lambda_{i's}(x,y,t)$ denotes some known physical realistic functions.

In this research work, our main goal is to develop a most powerful and highly efficient scheme viz., Bernstein polynomial operational matrix method to find the approximate numerical solution of the considered two-dimensional non-linear variable order reaction-diffusion equation with given initial and boundary conditions. The higher order convergence even for small grid approximation motivated us to use this highly accurate scheme. The Bernstein collocation scheme is very efficient tool in handling the non-linear variable order PDEs over other known scheme. The choice of collocation points in this scheme is very significant for the convergence of numerical scheme. The Newton iteration method are being used to solve the non-linear algebraic system of equations during the process. This proposed scheme is applied to many physical models to show the high convergence and efficiency of the scheme.
The present scientific manuscript is structured in the following manner; section 2 consists of the definition of variable order derivatives and its properties. Characteristics of Bernstein polynomials are discussed in section 3. The definition and properties of Kronecker production is provided in section 4. The unknown function is approximated in section 5. Section 6 consist of the derivation of variable order operational matrix. The convergence scrutiny of proposed numerical technique and approximation are given in section 7. Analysis of the concerned model, tabular and pictorial presentation of some examples are shown in section 8 followed by conclu-

sion section.

2. Variable Order Fractional Derivative

This part of the manuscript consists of definitions and some basic properties of variable order differentiation. Many definitions of the variable order fractional derivatives of any arbitrary order $\mu(x,y,t)$ are given viz., Caputo's fractional derivatives for variable order, Riemann-Liouville's derivatives for variable order and Grünwald-Letinkov's derivatives for variable order etc. The Caputo's definition for variable order derivatives $\mu(x,y,t)$ with respect to time and space is given as [22, 29, 34]:

$$\frac{d^{\mu(x,y,t)}\zeta(x,y,t)}{dt^{\mu(x,y,t)}} = \frac{1}{\Gamma(g-\mu(x,y,t))}\int_0^t (t-\rho)^{g-\mu(x,y,t)-1}\zeta^{(g)}(x,y,\rho)d\rho, \tag{3}$$

$$\frac{d^{\mu(x,y,t)}\zeta(x,y,t)}{dx^{\mu(x,y,t)}} = \frac{1}{\Gamma(g-\mu(x,y,t))}\int_0^x (x-\rho)^{g-\mu(x,y,t)-1}\zeta^{(g)}(\rho,y,t)d\rho, \tag{4}$$

and

$$\frac{d^{\mu(x,y,t)}\zeta(x,y,t)}{dy^{\mu(x,y,t)}} = \frac{1}{\Gamma(g-\mu(x,y,t))}\int_0^y (y-\rho)^{g-\mu(x,y,t)-1}\zeta^{(g)}(x,\rho,t)d\rho. \tag{5}$$

In the above definitions, g is the first integer number which is not less than $\mu(x,t)$, i.e., $g-1 < \mu(x,y,t) < g$ and $\Gamma(.)$ is Gamma function.

3. Bernstein Polynomial

The Bernstein polynomials are world wide utilized in many aspects of applied and engineering mathematics [9, 45]. The l^{th} degree Bernstein polynomials are defined in the unit interval [0,1] as:

$$B_{p,l}(x) = \binom{l}{p} x^p(1-x)^{l-p}, 0 \le p \le l. \tag{6}$$

As $0 \le x \le 1$, we can use Binomial expansion in the above equation, we have

$$B_{p,l}(x) = \binom{l}{p} x^p(\sum_{s=0}^{l-p}(-1)^s\binom{l-p}{s}x^s), 0 \le p \le l, \tag{7}$$

Or,

$$B_{p,l}(x) = \sum_{s=0}^{l-p}(-1)^s\binom{l}{p}\binom{l-p}{s}x^{p+s}, 0 \le p \le l. \tag{8}$$

The matrix form Bernstein polynomials can be expressed as

$$\vartheta(x) = MP_l(x), \tag{9}$$

where $\vartheta(x) = [B_{0,l}(x), B_{1,l}(x), ..., B_{l,l}(x)]^T$, $P_l(x) = [1, x, x^2, ..., x^l]^T$ and M is an upper triangular matrix as:

$$M = \begin{bmatrix} (-1)^0\binom{l}{0} & (-1)^1\binom{l}{0}\binom{l-0}{1-0} & \dots & \dots & \dots & (-1)^{m-0}\binom{l}{0}\binom{l-0}{1-0} \\ 0 & (-1)^0\binom{l}{1} & \dots & \dots & \dots & (-1)^{m-1}\binom{l}{1}\binom{l-1}{l-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & (-1)^0\binom{l}{i} & \dots & (-1)^{m-i}\binom{l}{i}\binom{l-i}{l-i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & (-1)^0\binom{l}{l} \end{bmatrix}. \tag{10}$$

In view of above equation, it is clear that matrix M is an invertible matrix i.e., $|M| \neq 0$. Few properties of Bernstein polynomials are given as:
(i) $B_{p,l}(x) \geq 0, \forall x \in [0, 1]$ i.e., Bernstein polynomials are always positive in their domain.
(ii) $B_{p,l}(1 - x) = B_{l-p,l}(x)$ and (iii) $B'_{p,l}(x) = l(B_{p-1,l-1}(x) - B_{p,l-1}(x))$.

4. Properties of Kronecker Product

Some important properties of Kronecker Product will be discussed in this part of the article [6, 14]. Moreover, we try to provide some overview of the applications and properties of the Kronecker Product, which is denoted by the symbol $\otimes$, an operation between two arbitrary size matrices.

Definition.

For matrix A of order $p \times q$ and matrix B of order $m \times n$, the Kronecker Product is again a block matrix of order $pm \times qn$, which is denoted and defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1q}B \\ a_{21}B & a_{22}B & \dots & a_{21}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{p1}B & a_{p2}B & \dots & a_{pq}B \end{bmatrix}. \tag{11}$$

Moreover, If A and B represent some linear transformations on vector spaces i.e., $W_1 \rightarrow V_1$ and $W_2 \rightarrow V_2$ respectively, then $A \otimes B$ represents the tensor product of these two maps, $W_1 \otimes W_2 \rightarrow V_1 \otimes V_2$.

Basic Properties.

It is well known that Kronecker product is a special case of tensor product, so it can satisfy the property like associativity and bi-linearity. Some important properties of this product is given below

$$I \otimes (J + K) = I \otimes J + I \otimes K, \tag{12}$$

$$(I + J) \otimes K = I \otimes K + J \otimes K, \tag{13}$$

$$(kI \otimes J) = (I \otimes kJ) = k(I \otimes J), \tag{14}$$

$$(I \otimes J) \otimes K = I \otimes (J \otimes K), \tag{15}$$

where I, J and K are any matrices and k is a scalar.

5. Approximation of the Unknown Function $\zeta(x, y, t)$

The collection of Bernstein polynomials forms a complete basis in the Hilbert space $L^2[0, 1]$ therefore each function $\zeta(x) \in L^2[0, 1]$ can be expressed in terms of Bernstein polynomials as:

$$\zeta(x) \simeq \zeta_r(x) = \sum_{g=0}^{r} b_g B_{g,r}(x) = B^T.\vartheta(x), \tag{16}$$

where the unknown constants matrix $B^T = [b_g]$ is known as Bernstein coefficients.
Again a function of two variable $\zeta(x, t) \in L^2[0, 1]$ can be written in terms of Bernstein polynomials as:

$$\zeta(x, t) \simeq \zeta_r(x, t) = \sum_{g=0}^{r}\sum_{h=0}^{r} b_{g,h} B_{g,r}(x) B_{h,r}(t) = \vartheta(x)^T.B.\vartheta(t), \tag{17}$$

here the undetermined constants matrix $B = [b_{g,h}]$ is known as Bernstein coefficients.
Similarly, one can approximate the two-dimensional function $\zeta(x, y, t) \in L^2[0, 1]$ in terms of Bernstein polynomial as:

$$\begin{aligned} \zeta(x, y, t) \simeq \zeta_r(x, y, t) &= \sum_{g=0}^{r}\sum_{h=0}^{r}\sum_{i=0}^{r} b_{g,h,i} B_{g,r}(x) B_{h,r}(y) B_{i,r}(t) \\ &= \vartheta(t)^T.B.\vartheta(x) \otimes \vartheta(y). \end{aligned} \tag{18}$$

here the undetermined constants matrix $B = [b_{g,h,i}]$ is known as Bernstein coefficients. These unknowns can be determine by using the given initial and boundary conditions.

6. Variable order operational matrix

The variable order operational matrix will be deduced in this section of the article. In view of equation (9), the differentiation of vector $\vartheta(t)$ is

$$\frac{\partial^{\mu(x,y,t)}\vartheta(t)}{\partial t^{\mu(x,y,t)}} = \frac{\partial^{\mu(x,y,t)} M.P_r(t)}{\partial t^{\mu(x,y,t)}} = M.\frac{\partial^{\mu(x,y,t)}}{\partial t^{\mu(x,y,t)}} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{g-1} \\ t^g \\ \vdots \\ t^r \end{bmatrix}, \tag{19}$$

here $g-1 \leq \mu(x,y,t) \leq g$, $g = \lceil \mu(x,y,t) \rceil$ and $g < r$.

$$\frac{\partial^{\mu(x,y,t)}\vartheta(t)}{\partial t^{\mu(x,y,t)}} = M. \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{\Gamma(g+1).t^{g-\mu(x,y,t)}}{\Gamma(g+1-\mu(x,y,t))} \\ \vdots \\ \frac{\Gamma(r+1).t^{r-\mu(x,y,t)}}{\Gamma(r+1-\mu(x,y,t))} \end{bmatrix}. \tag{20}$$

In matrix form it can also be represented as

$$\frac{\partial^{\mu(x,y,t)}\vartheta(t)}{\partial t^{\mu(x,y,t)}} = M. \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{\Gamma(g+1).t^{-\mu(x,y,t)}}{\Gamma(g+1-\mu(x,y,t))} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \frac{\Gamma(r+1).t^{-\mu(x,y,t)}}{\Gamma(r+1-\mu(x,y,t))} \end{bmatrix} . \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{g-1} \\ t^g \\ \vdots \\ t^r \end{bmatrix}. \tag{21}$$

We can also write this equation as

$$\frac{\partial^{\mu(x,y,t)}\vartheta(t)}{\partial t^{\mu(x,y,t)}} = M.\varOmega.P_r(t), \tag{22}$$

here $\varOmega$ is:

$$\varOmega = [a_{pq}]_{(r+1)\times(r+1)} = \begin{cases} 0, & \text{else}, \\ \frac{\Gamma(g+1).t^{-\mu(x,y,t)}}{\Gamma(g+1-\mu(x,y,t))}, \text{when } p = q \geq g. \end{cases} \tag{23}$$

From equation (9) and(22), we have

$$\frac{\partial^{\mu(x,y,t)}\vartheta(t)}{\partial t^{\mu(x,y,t)}} = M.\varOmega.M^{-1}.\vartheta(t), \tag{24}$$

here $M.\varOmega.M^{-1}$ is an operational matrix of fractional order differentiation w.r.to temporal variable. In the same way one can find the same for variable order differentiation w.r.to spatial variable x and y. Collocating the considered model (1) under the consideration of stated initial and boundary conditions (2), we get a non-linear fractional order differentiation w.r.to algebraic equation. On solving this non-linear algebraic equation, we find the the matrix B in the approximation (18).

7. Verification of numerical stability and consistency

Theorem Let the function $\zeta(x,y,t) : [0,1] \times [0,1] \times [0,1] \longrightarrow \mathbb{R}$ is a r times continuously differentiable function, i.e. $\zeta \in C^r[0,1]^3$, and let
$\mathbb{X} = \text{Span}[B_{p,r}(x)B_{q,r}(y)B_{s,r}(t), p,q,s = 0,1,\cdots,r]$ be a linear space. Let $\vartheta(t)^T.B.\vartheta(x) \otimes \vartheta(y)$ is best approximation of the function $\zeta(x,y,t)$ out of $\mathbb{X}$ then the mean approximation of error $\parallel \zeta(x,y,t) - \vartheta(t)^T.B.\vartheta(x) \otimes \vartheta(y) \parallel_2$ is bounded.

Proof: For the function $\zeta(x,y,t)$, the Taylor series approximation in three variables in the neighborhood of a point (x_0,y_0,t_0) is given by

$$\begin{aligned} \zeta_r(x,y,t) = \zeta(x_0,y_0,t_0) &+ \zeta_x(x_0,y_0,t_0)(x-x_0) + \zeta_y(x_0,y_0,t_0)(y-y_0) \\ &+ \zeta_t(x_0,y_0,t_0)(t-t_0) + \cdots \end{aligned} \tag{25}$$

now we can write

$$\begin{aligned} \mid \zeta(x,y,t) - \zeta_r(x,y,t) \mid = \mid \zeta(x_0,y_0,t_0) + \sum_{r=1}^{\infty}(\sum_{\substack{u,v,w \\ (u+v+w=r)}} \frac{1}{u!v!w!} \\ \zeta^r_{x^u y^v t^w}(x_0,y_0,t_0)(x-x_0)^u(y-y_0)^v(t-t_0)^w) \mid . \end{aligned} \tag{26}$$

Let $\vartheta(t)^T.B.\vartheta(x) \otimes \vartheta(y)$ be the best approximation of $\zeta(x,y,t)$ then we can write

$$\begin{aligned} \parallel \zeta(x,y,t) - \vartheta(t)^T.B.\vartheta(x) \otimes \vartheta(y) \parallel_2^2 &\leq \parallel \zeta(x,y,t) - \zeta_r(x,y,t) \parallel_2^2 \\ &= \int_0^1 \int_0^1 \int_0^1 \mid \zeta(x,y,t) - \zeta_r(x,y,t) \mid^2 dxdydt, \end{aligned} \tag{27}$$

Thus in view of equation (26), we have

$$\begin{aligned} \parallel \zeta(x,y,t) - \vartheta(t)^T.B.\vartheta(x) \otimes \vartheta(y) \parallel_2^2 \leq \int_0^1 \int_0^1 \int_0^1 \mid \zeta(x_0,y_0,t_0) \\ + \sum_{r=1}^{\infty}(\sum_{\substack{u,v,w \\ (u+v+w=r)}} \frac{1}{u!v!w!} \zeta^r_{x^u y^v t^w}(x_0,y_0,t_0)(x-x_0)^u(y-y_0)^v(t-t_0)^w) \mid dxdydt. \end{aligned} \tag{28}$$

It is assumed that $\zeta^r_{x^u y^v t^w}(x,y,t)$ is r-times continuously-differentiable map, so there exist a constant S_1 such that

$$\max_{0 \leq x,y,t \leq 1} \zeta^r_{x^u y^v t^w}(x,y,t) \leq S_1, u+v+w = r; \tag{29}$$

we have

$$\begin{aligned} \parallel \zeta(x,y,t) - \vartheta(t)^T.B.\vartheta(x) \otimes \vartheta(y) \parallel_2^2 \leq \int_0^1 \int_0^1 \int_0^1 \mid \zeta(x_0,y_0,t_0) \\ + \sum_{r=1}^{\infty}(\sum_{\substack{u,v,w \\ (u+v+w=r)}} \frac{1}{u!v!w!} S_1(x-x_0)^u(y-y_0)^v(t-t_0)^w) \mid dxdydt. \end{aligned} \tag{30}$$

Now let $S_2 = max[\frac{1}{u!v!w!}; u+v+w=r]$. Then from equation (30), we can write

$$\| \zeta(x,y,t) - \vartheta(t)^T.B.\vartheta(x)\otimes\vartheta(y) \|_2^2 \leq \int_0^1\int_0^1\int_0^1 \mid \zeta(x_0,y_0,t_0) + \sum_{r=1}^{\infty}(\sum_{\substack{u,v,w\\(u+v+w=r)}} S_2S_1(x-x_0)^u(y-y_0)^v(t-t_0)^w) \mid dxdydt. \tag{31}$$

Let τ_j be the zeros of the Bernstein Polynomials, for $(x-x_0)^u$ we can found the following bounds by using the transformation $x=\frac{\tau+1}{2}$ among the intervals [-1,1] and [0,1] as:

$$\min_{0\leq x_j\leq 1}\max_{0\leq x\leq 1} \mid (x-x_0)^u \mid = \min_{0\leq \tau_j\leq 1}\max_{0\leq \tau\leq 1} \mid (\frac{\tau-\tau_j}{2})^u \mid = \frac{1}{2^{2u+1}}. \tag{32}$$

Thus form the equation (31), we can write

$$\| \zeta(x,y,t) - \vartheta(t)^T.B.\vartheta(x)\otimes\vartheta(y) \|_2^2 \leq \int_0^1\int_0^1\int_0^1 \mid \zeta(x_0,y_0,t_0) + \sum_{r=1}^{\infty}(\sum_{\substack{u,v,w\\(u+v+w=r)}} S_2S_1\frac{1}{2^{2u+1}}\frac{1}{2^{2v+1}}\frac{1}{2^{2w+1}} \mid dxdydt. \tag{33}$$

Or,

$$\| \zeta(x,y,t) - \vartheta(t)^T.B.\vartheta(x)\otimes\vartheta(y) \|_2^2 \leq \int_0^1\int_0^1\int_0^1 \mid \zeta(x_0,y_0,t_0) + S_2S_1\sum_{r=1}^{\infty}\frac{1}{2^{2r+3}} \mid dxdydt. \tag{34}$$

In the above inequality all the terms on the right hand side are constants. Therefore the error of approximation $\| \zeta(x,y,t) - \vartheta(t)^T.B.\vartheta(x)\otimes\vartheta(y) \|_2$ is bounded. Hence we get our required proof.

8. Predictive Simulation and Error Estimation

In this part of research work the Bernstein polynomials operational matrix numerical technique has been used on some non-linear two-dimensional variable order partial differential equations and further we compared the obtained numerical result with its exact solutions and solutions from some other known numerical methods to show the authenticity and accuracy of the proposed numerical technique.
To validate the authenticity and efficiency by error estimation, the RMS error, L_∞ and L_2 errors have been taken in to consideration according as [37].

Example 1: Consider the following two-dimensional variable order reaction-diffusion equation which is a particular case of our concerned model (1) with $p_1(x,y,t)=0, p_2(x,y,t)=1, p_3(x,y,t)=1, p_4(x,y,t)=0$ and $p_5(x,y,t)=\frac{2t^{2-\mu_1(x,y,t)}}{\Gamma(3-\mu_1(x,y,t))}+2x+2y-4$ then we have the following two-dimensional variable order PDE as:

$$\frac{\partial^{\mu_1(x,y,t)}\zeta(x,y,t)}{\partial t^{\mu_1(x,y,t)}} = \frac{\partial^2\zeta(x,y,t)}{\partial x^2} + \frac{\partial^2\zeta(x,y,t)}{\partial y^2} - \frac{\partial\zeta(x,y,t)}{\partial x} - \frac{\partial\zeta(x,y,t)}{\partial y} + \frac{2t^{2-\mu_1(x,y,t)}}{\Gamma(3-\mu_1(x,y,t))} + 2x + 2y - 4, \tag{35}$$

The exact solution of this model is $\zeta(x,y,t) = x^2+y^2+t^2$. The initial and boundary conditions for problem can be derived from the exact solution. The graph of absolute error between the obtained numerical solution with the existing exact solution is shown through the Fig. 1 for $\mu_1(x,y,t) = 0.8 - 0.1\cos(xt)\sin x - 0.1\cos(yt)\sin y$ at $t = 0.05$. The comparison of RMS error and L_∞ error between obtained numerical solution and solution obtained by the methods given in [28, 31] is shown in the Table 1.

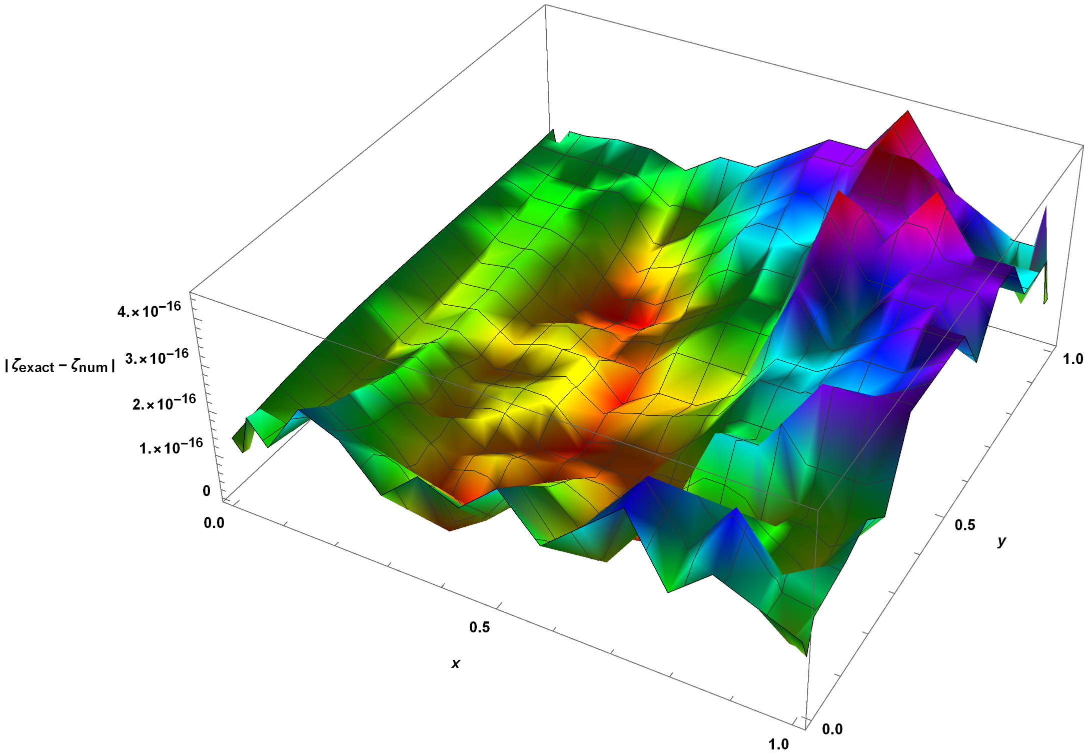

Figure 1: Error graph between the numerical and exact solutions of $\zeta(x,y,t)$ vs. y and x.

Table 1: Comparison of RMS and L_∞ norm error for $\zeta(x,y,t)$ at $t=1$

	Proposed Method		Meerschaert [31]		Liu [28]	
μ	L_∞	RMS	L_∞	RMS	L_∞	RMS
0.5	4.7636×10^{-11}	1.2311×10^{-11}	3.2296×10^{-5}	1.6596×10^{-5}	4.1163×10^{-5}	3.1388×10^{-5}
0.8	1.9672×10^{-12}	1.2983×10^{-12}	2.1712×10^{-4}	1.1143×10^{-4}	1.2445×10^{-4}	7.2997×10^{-5}
$\mu(x,y,t)$	7.7349×10^{-12}	3.0374×10^{-12}	1.3394×10^{-4}	6.9272×10^{-5}	8.5857×10^{-5}	5.3835×10^{-5}

Fig. 1 is the graphical exhibition of absolute error between the exact and numerical solution which clearly shows that the obtained result by numerical technique is very close to exact solution for small values of $r = 5$. Table 1 shows that the errors in the computation of numerical solution through the proposed scheme.

Example 2: Consider the following non-linear variable order reaction-diffusion equation which is a particular case of our concerned model (1) with $p_1(x,y,t) = 0, p_2(x,y,t) = 1, p_3(x,y,t) = [1\ 1]^T$ and $p_4(x,y,t) = \sin(2\zeta(x,y,t))$ then we have the following two-dimensional variable order PDE as:

$$\frac{\partial^{\mu_1(x,y,t)}\zeta(x,y,t)}{\partial t^{\mu_1(x,y,t)}} = \frac{\partial^2\zeta(x,y,t)}{\partial x^2} + \frac{\partial^2\zeta(x,y,t)}{\partial y^2} - \frac{\partial\zeta(x,y,t)}{\partial x} - \frac{\partial\zeta(x,y,t)}{\partial y} + \sin(2\zeta(x,y,t)) + p_5(x,y,t), \tag{36}$$

The exact solution of this model is $\zeta(x,y,t) = x^2 + y^2 + t^2$ for suitable value of $p_5(x,y,t)$. The initial and boundary conditions for problem can be derived from the exact solution. The graph of absolute error between the obtained numerical solution with the existing exact solution is shown through the Fig. 2 for $\mu_1(x,y,t) = 1 - 0.75exp(-(xyt)^2)$ at $t = 0.05$. The comparison of RMS error, L_2 error and L_∞ error between obtained numerical solution and solution obtained by the method given in [18] is shown in the Table 2.

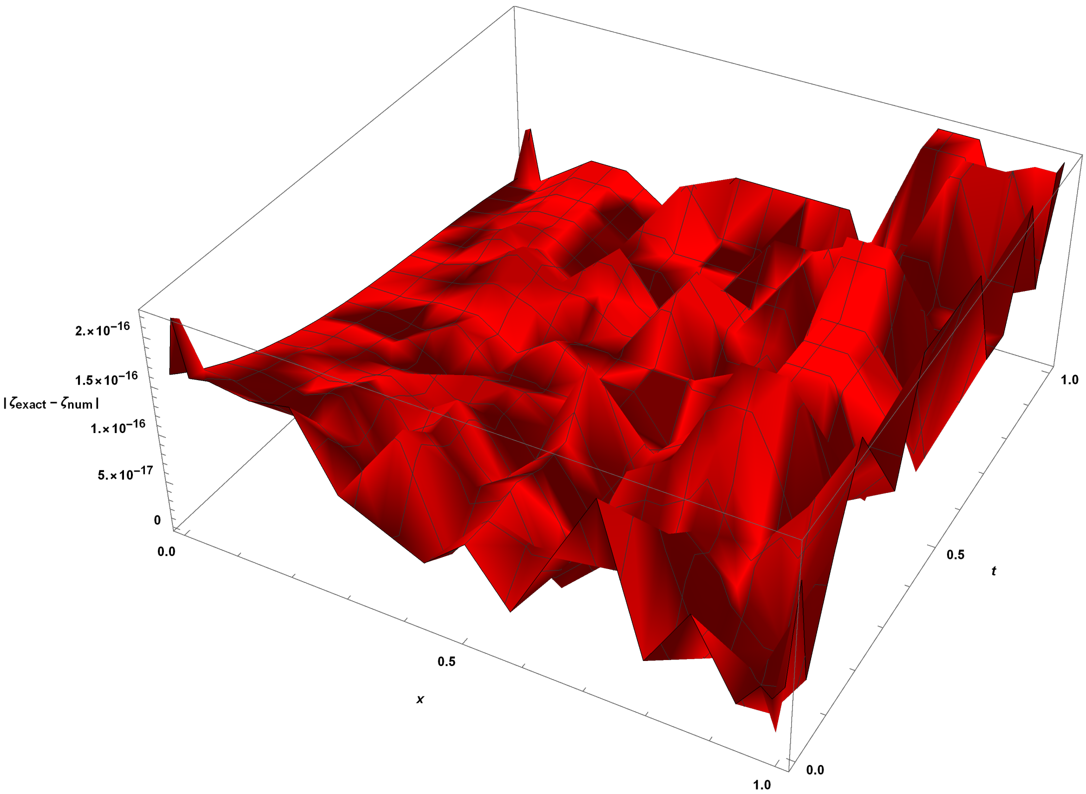

Figure 2: Error graph between the numerical and exact solutions of $\zeta(x,y,t)$ vs. y and x.

Table 2: Comparison of RMS, L_2 and L_∞ norm error for $\zeta(x, y, t)$ at $t = 1$

	Proposed Method			Hosseininia [18]		
t	L_∞	L_2	RMS	L_∞	L_2	RMS
0.2	2.9223 $\times 10^{-14}$	4.2837 $\times 10^{-14}$	3.2398 $\times 10^{-15}$	1.2952 $\times 10^{-4}$	3.6041 $\times 10^{-4}$	4.5051 $\times 10^{-5}$
0.4	8.8533 $\times 10^{-14}$	9.9827 $\times 10^{-14}$	4.2032 $\times 10^{-15}$	1.7426 $\times 10^{-4}$	5.3018 $\times 10^{-4}$	6.6273 $\times 10^{-5}$
0.6	5.0973 $\times 10^{-13}$	7.9843 $\times 10^{-13}$	2.1274 $\times 10^{-14}$	3.0199 $\times 10^{-4}$	8.9481 $\times 10^{-4}$	1.1185 $\times 10^{-4}$
0.8	1.0983 $\times 10^{-12}$	1.7632 $\times 10^{-12}$	3.0877 $\times 10^{-14}$	6.8715 $\times 10^{-4}$	2.1749 $\times 10^{-3}$	2.7187 $\times 10^{-4}$
1	6.2920 $\times 10^{-12}$	6.9837 $\times 10^{-12}$	3.0928 $\times 10^{-13}$	1.0291 $\times 10^{-3}$	3.4972 $\times 10^{-3}$	4.3715 $\times 10^{-4}$

Fig. 2 is the graphical exhibition of absolute error between the exact and numerical solution which clearly shows that the obtained result by numerical technique is very close to exact solution for small values of $r = 5$. Table 2 shows that the errors in the computation of numerical solution through the proposed scheme.

Example 3: Consider the following non-linear variable order reaction-diffusion equation which is a particular case of our concerned model (1) with $p_1(x, y, t) = 0, p_2(x, y, t) = x^2 + y^2 + t + 1, p_3(x, y, t) = [t^2e^{x+y}\ t^2e^{x+y}]^T$ and $p_4(x, y, t) = \sin(2\zeta(x, y, t))$ then we have the following two-dimensional variable order PDE as:

$$\begin{aligned}\frac{\partial^{\mu_1(x,y,t)}\zeta(x,y,t)}{\partial t^{\mu_1(x,y,t)}} &= (x^2+y^2+t+1)\left(\frac{\partial^2\zeta(x,y,t)}{\partial x^2}+\frac{\partial^2\zeta(x,y,t)}{\partial y^2}\right)\\ &- t^2e^{x+y}\left(\frac{\partial\zeta(x,y,t)}{\partial x}+\frac{\partial\zeta(x,y,t)}{\partial y}\right)+\sin(2\zeta(x,y,t))+p_5(x,y,t),\end{aligned} \tag{37}$$

The exact solution of this model is $\zeta(x, y, t) = x^2 + y^2 + t^2$ for suitable value of $p_5(x, y, t)$. The initial and boundary conditions for problem can be derived from the exact solution. The graph of absolute error between the obtained numerical solution with the existing exact solution is shown through the Fig. 3 for $\mu_1(x, y, t) = 1 - 0.75exp(-(xyt)^2)$ at $t = 0.05$. The comparison of RMS error, L_2 error and L_∞ error between obtained numerical solution and solution obtained by the method given in [18] is shown in the Table 3.

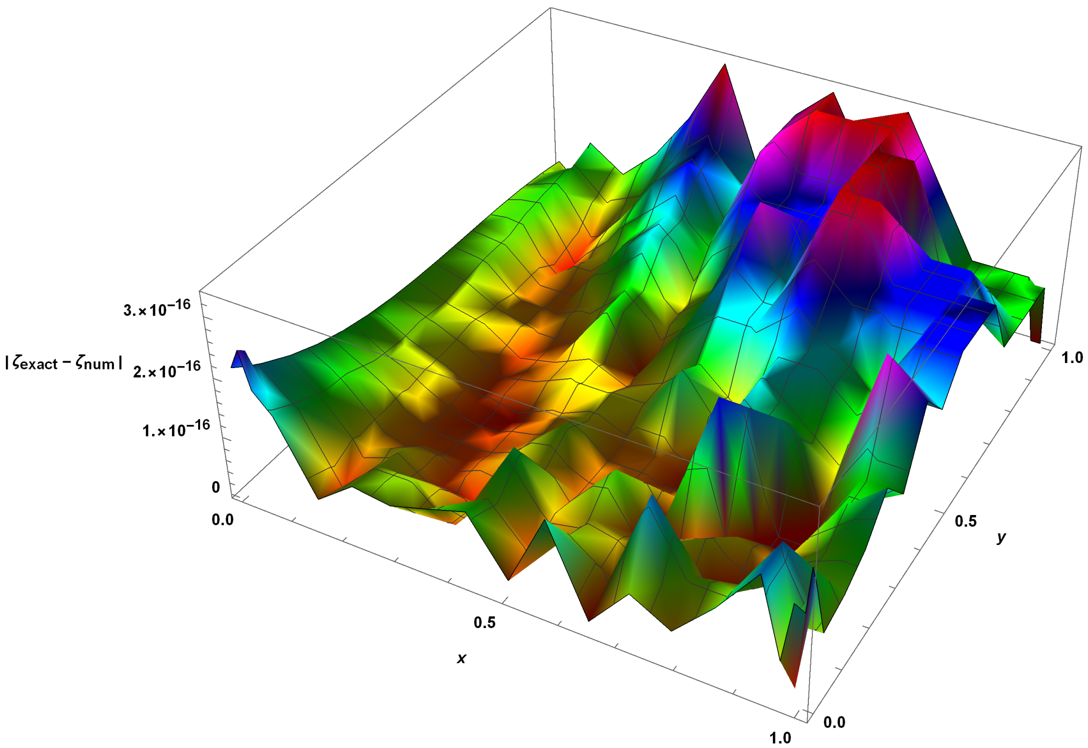

Figure 3: Error graph between the numerical and exact solutions of $\zeta(x,y,t)$ vs. y and x.

Table 3: Comparison of RMS, L_2 and L_∞ norm error for $\zeta(x,y,t)$ at $t=1$

	Proposed Method			Hosseininia [18]		
t	L_∞	L_2	RMS	L_∞	L_2	RMS
0.2	5.2038×10^{-14}	5.4183×10^{-14}	2.4738×10^{-15}	8.5523×10^{-5}	2.4503×10^{-4}	3.0629×10^{-5}
0.4	1.0023×10^{-14}	2.1434×10^{-14}	3.2782×10^{-15}	1.2400×10^{-4}	3.0871×10^{-4}	3.8589×10^{-5}
0.6	2.0239×10^{-13}	2.9034×10^{-13}	4.3842×10^{-15}	1.2427×10^{-4}	3.8599×10^{-4}	4.8250×10^{-5}
0.8	7.8432×10^{-12}	8.9083×10^{-12}	4.1321×10^{-14}	3.0315×10^{-4}	9.5822×10^{-4}	1.1978×10^{-4}
1	3.1311×10^{-12}	5.3422×10^{-12}	5.2322×10^{-13}	4.5133×10^{-4}	1.5505×10^{-3}	1.9381×10^{-4}

Fig. 3 is the graphical exhibition of absolute error between the exact and numerical solution which clearly shows that the obtained result by numerical technique is very close to exact solution for small values of $r=5$. Table 3 shows that the errors in the computation of numerical solution through the proposed scheme.

Example 4: Consider the following non-linear variable order two-dimensional reaction-diffusion equation which is a particular case of our concerned model (1) with $p_1(x,y,t)=0, p_2(x,y,t)=10, p_3(x,y,t)=[4\ 4]^T$ and $p_4(x,y,t)=\zeta^2(x,y,t)$ then we have the following two-dimensional variable order PDE as:

$$\frac{\partial^{\mu_1(x,y,t)}\zeta(x,y,t)}{\partial t^{\mu_1(x,y,t)}} = 10\frac{\partial^2\zeta(x,y,t)}{\partial x^2} + 10\frac{\partial^2\zeta(x,y,t)}{\partial y^2} - 4\frac{\partial\zeta(x,y,t)}{\partial x} - 4\frac{\partial\zeta(x,y,t)}{\partial y} + \zeta^2(x,y,t) + p_5(x,y,t), \tag{38}$$

The exact solution of this model is $\zeta(x,y,t) = (\cos x + \cos y)t^{3+\mu_1(x,y,t)}$ for suitable value of $p_5(x,y,t)$. The initial and boundary conditions for problem can be derived from the exact solution. The graph of absolute error between the obtained numerical solution with the existing exact solution is shown through the Fig. 4 for $\mu_1(x,y,t) = 0.8 - 0.1\cos(xt)\sin x - 0.1\cos(yt)\sin y$ at $t = 0.05$. The comparison of RMS error, L_2 error and L_∞ error between obtained numerical solution and solution obtained by the method given in [18] is shown in the Table 4.

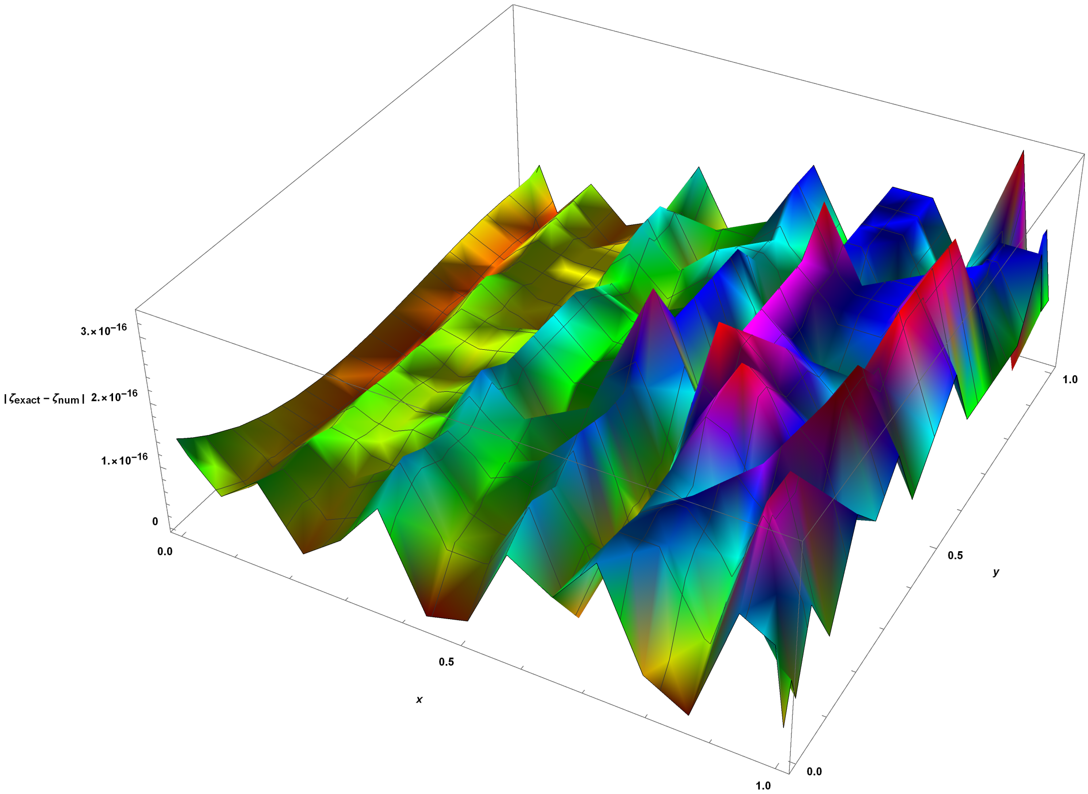

Figure 4: Error graph between the numerical and exact solutions of $\zeta(x,y,t)$ vs. y and x.

Table 4: Comparison of RMS, L_2 and L_∞ norm error for $\zeta(x,y,t)$ at $t=1$

	Proposed Method			Hosseininia [18]		
t	L_∞	L_2	RMS	L_∞	L_2	RMS
0.2	1.2398×10^{-14}	3.9374×10^{-14}	1.0932×10^{-15}	1.0568×10^{-4}	3.6423×10^{-4}	4.3279×10^{-5}
0.4	3.2322×10^{-14}	4.0804×10^{-14}	3.0900×10^{-15}	9.1658×10^{-4}	3.0362×10^{-3}	3.7952×10^{-4}
0.6	3.2038×10^{-13}	7.1232×10^{-13}	5.8699×10^{-14}	2.6145×10^{-3}	8.7059×10^{-3}	1.0882×10^{-3}
0.8	1.1223×10^{-12}	7.7689×10^{-12}	9.2298×10^{-14}	3.7758×10^{-3}	1.2609×10^{-2}	1.5762×10^{-3}
1	5.6472×10^{-12}	6.0862×10^{-12}	1.2634×10^{-13}	6.2003×10^{-5}	2.2556×10^{-4}	2.8196×10^{-5}

Fig. 4 is the graphical exhibition of absolute error between the exact and numerical solution which clearly shows that the obtained result by numerical technique is very close to exact solution for small values of $r = 5$. Table 4 shows that the errors in the computation of numerical solution through the proposed scheme.

Example 5: Consider the following non-linear variable order two-dimensional reaction-diffusion equation which is a particular case of our concerned model (1) with $p_1(x,y,t) = 0, p_2(x,y,t) = x + y + 1, p_3(x,y,t) = [t\sin x + 2\sin y \;\; t\sin y]^T$ and $p_4(x,y,t) = \zeta^2(x,y,t)$ then we have the following two-dimensional variable order PDE as:

$$\frac{\partial^{\mu_1(x,y,t)}\zeta(x,y,t)}{\partial t^{\mu_1(x,y,t)}} = (x+y+1)(\frac{\partial^2\zeta(x,y,t)}{\partial x^2} + \frac{\partial^2\zeta(x,y,t)}{\partial y^2})- \\ (t\sin x + 2\sin y)\frac{\partial\zeta(x,y,t)}{\partial x} - t\sin y\frac{\partial\zeta(x,y,t)}{\partial y} + \zeta^2(x,y,t) + p_5(x,y,t), \tag{39}$$

The exact solution of this model is $\zeta(x,y,t) = (\cos x + \cos y)t^{3+\mu_1(x,y,t)}$ for suitable value of $p_5(x,y,t)$. The initial and boundary conditions for problem can be derived from the exact solution. The graph of absolute error between the obtained numerical solution with the existing exact solution is shown through the Fig. 5 for $\mu_1(x,y,t) = 0.8-0.1\cos(xt)\sin x-0.1\cos(yt)\sin y$ at $t = 0.05$. The comparison of RMS error, L_2 error and L_∞ error between obtained numerical solution and solution obtained by the method given in [18] is shown in the Table 5.

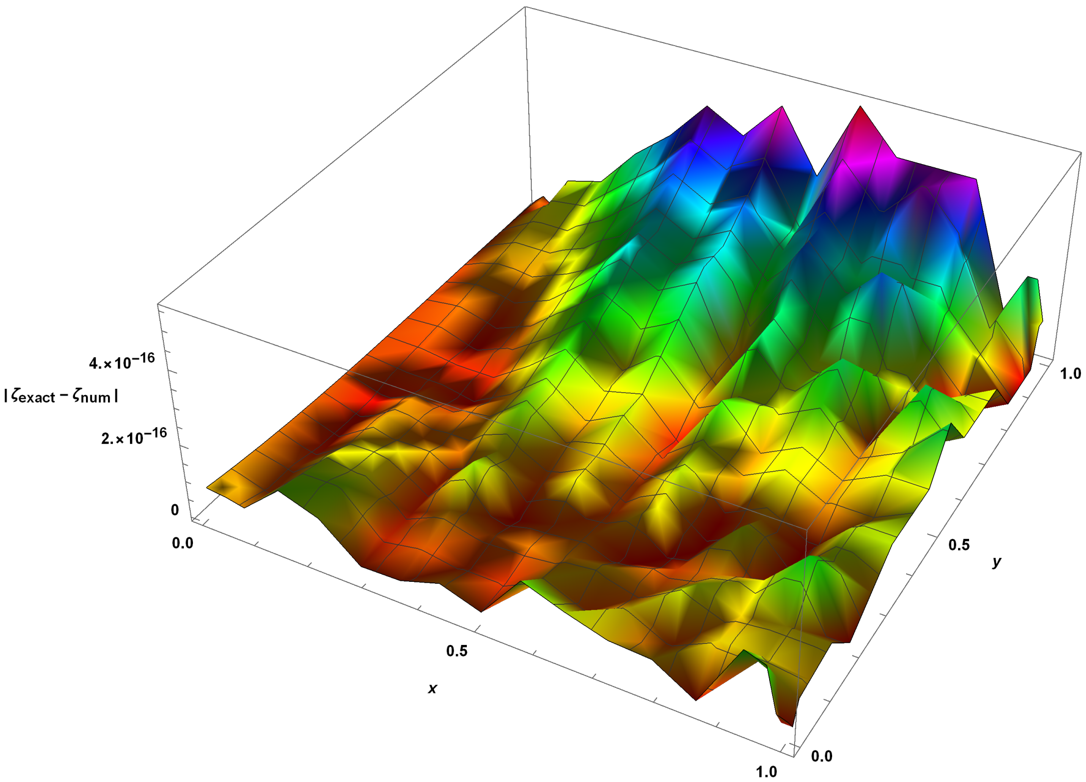

Figure 5: Error graph between the numerical and exact solutions of $\zeta(x,y,t)$ vs. y and x.

Table 5: Comparison of RMS, L_2 and L_∞ norm error for $\zeta(x,y,t)$ at $t=1$

	Proposed Method			Hosseininia [18]		
t	L_∞	L_2	RMS	L_∞	L_2	RMS
0.2	5.3522×10^{-14}	6.3094×10^{-14}	4.9374×10^{-15}	5.9901×10^{-5}	1.8595×10^{-4}	2.3244×10^{-5}
0.4	2.0833×10^{-14}	2.9724×10^{-14}	1.2390×10^{-15}	6.9412×10^{-5}	2.2458×10^{-3}	2.8072×10^{-4}
0.6	1.0032×10^{-13}	4.5384×10^{-13}	9.0026×10^{-14}	2.3356×10^{-3}	7.7032×10^{-3}	6.6290×10^{-4}
0.8	5.0343×10^{-12}	6.0933×10^{-12}	3.2827×10^{-14}	4.2359×10^{-3}	1.4335×10^{-2}	1.7919×10^{-3}
1	2.3092×10^{-12}	3.6493×10^{-12}	3.4833×10^{-13}	3.7488×10^{-3}	1.2351×10^{-2}	1.5439×10^{-3}

Fig. 5 is the graphical exhibition of absolute error between the exact and numerical solution which clearly shows that the obtained result by numerical technique is very close to exact solution for small values of $r = 5$. Table 5 shows that the errors in the computation of numerical solution through the proposed scheme.

9. Conclusion

In this research work three main objectives have been achieved. The first one is to derive the Bernstein operational matrix for variable order derivatives for the spacial and temporal variable. Next is the finding of solute concentration $\zeta(x, y, t)$ for our considered reaction-diffusion equation of two-dimensional non-linear variable order by using the strong and efficient tool Bernstein operational matrix together having background discussion of variable order fractional derivatives. The most important parts of the present study are the to discuss stability analysis, finding the error bounds and the exhibition of the order of convergence during the validation of discussed tool while using it on the arising complex problems having analytical solutions. The authors are optimist that the numerical solution by the proposed algorithm can be extended in a wide range i.e., for solving 3D non-linear reaction-diffusion equation in time-space fractional variable order problems which is the topic of our future research.

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