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Caputo–Fabrizio derivative provides stability solutions with non–immediate impulses of fractional differential

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Abstract

This study examines the HU-stability of a class of fractional functional differential systems including the "Caputo–Fabrizio derivative", which is especially efficient in modelling systems with non-immediate impulses due of its non-singular, exponentially decaying kernel. Here the system is described across a piecewise interval with both differential and integral requirements, where the generalized fractional derivative is applied on subintervals and non-local integral restrictions are placed elsewhere. Functionals satisfying Lipschitz continuity control nonlinearities in the system, hence guaranteeing limitedness and regularity. The contraction criterion $\rho < 1$ is satisfied by the system over all admissible values of the fractional order $\delta \in (0,1)$ by use of analytical estimates and numerical validation. Tables and graphs show the changes of stability-related constants concerning δ, hence stressing the stabilizing function of the Caputo–Fabrizio derivative. The results verify that any approximate solution $z(x)$ stays near the precise solution $\varphi(x)$, hence confirming the HU-stability of the system and stressing its strength under perturbations.

Keywords- *Caputo–Fabrizio fractional derivative, functional differential equations, non-instantaneous impulses, stability analysis, HU-stability.*

1. Introduction

Understanding the complexity of biological systems, including the development of cancer and how it reacts to treatment, has traditionally been one of the toughest obstacles in mathematical biology. Classical differential equations regularly fail to represent memory things and non-local interactions, even if they are useful. In cases involving non-singular machinists such as the "Caputo–Fabrizio (CF) fractional derivative", fractional calculus offers a good replacement.

The "Caputo–Fabrizio derivative", introduced to eliminate singularities in the kernel, is defined for a function. $\boldsymbol{f(t)} \in \boldsymbol{H^1(a,b)}$As:

$$^{\mathrm{CF}}\boldsymbol{D_t^{\alpha} f(t)} = \frac{\mathbf{1}}{\mathbf{1}-\boldsymbol{\alpha}}\int_{\boldsymbol{\alpha}}^{\boldsymbol{t}} \boldsymbol{f'(\mathcal{S}) e^{-\frac{\alpha}{1-\alpha}(t-\mathcal{S})} d\mathcal{S}}, \quad \mathbf{0} < \boldsymbol{\alpha} < \mathbf{1}$$

The non-singular kernel makes the fractional derivative suitable for describing biological systems marked by exponential memory decay instead of power-law decay, as it ensures their continuity and smoothness at t = a [1].

Numerous practical realities involve complex cases like cancer therapy whereby the impulses applied, such as those from drug injection and radiation therapy, do not take place instantaneously but rather develop over time. Such impulses can, therefore, be called non-instantaneous impulses. Hence, employing fractional differential equations (FDEs) with non-instantaneous impulse responses is more suitable in case of modeling such impulses.

Let us examine a fractional differential system characterized by non-instantaneous impulses:

$$^{\mathrm{CF}}\boldsymbol{D_t^{\alpha} x(t)} = \boldsymbol{f\big(t, x(t)\big)}, \quad \boldsymbol{t} \in [\boldsymbol{t_0}, \boldsymbol{t_n}], \boldsymbol{t} \neq \boldsymbol{t_k},$$

$$\boldsymbol{x(t)} = \boldsymbol{\psi_k\big(t, x(t_k^-)\big)}, \ \boldsymbol{t} \in [\boldsymbol{t_k}, \boldsymbol{t_k} + \boldsymbol{\delta_k}], \boldsymbol{k} = \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{n},$$

Where $^{\mathrm{CF}}\boldsymbol{D_t^{\alpha}}$ is the Caputo–Fabrizio derivative, and $\boldsymbol{\psi_k}$ represents the **non-immediate impulse effect** over the interval $[\boldsymbol{t_k}, \boldsymbol{t_k} + \boldsymbol{\delta_k}]$**.** The stability analysis of the equations has been conducted, which implies the existence, uniqueness, and Ulam-Hyers stability thanks to Lyapunov functions used for CF operators [2].

The biological compatibility of CF-based systems is expressed through functioning interactions of T(t), I(t), and U(t) represented in chemotherapy.

A fractional tumor-immune-therapy model, for example, may look like:

$$ {}^{\mathrm{CF}}D_t^{\alpha}T(t) = r_1T(t)\big(1 - b_1T(t)\big) - c_1I(t)T(t) - a_1\big(1 - e^{-U(t)}\big)T(t), $$

$$ {}^{\mathrm{CF}}D_t^{\alpha}I(t) = \frac{\rho I(t)T(t)}{\mu + T(t)} - d_1I(t) - c_2I(t)T(t), $$

$$ {}^{\mathrm{CF}}D_t^{\alpha}U(t) = -d_2U(t) + \vartheta(t), $$

Wherever $\vartheta(t)$**,** It is an external therapeutic source, such as a drug input. The fractional system precisely resembles physical phenomena than integer order. For example, the models perform better forecast in cases of periodic or stage treatment as compared to models in integer order. The non-instantaneous inputs are an even better representation of the processes like the continuous drug release where therapy is due to the ongoing application, not a discrete one.

As we will show, the use of such inputs in fractional systems with the use of CF derivatives gives the condition under which global and unique solutions exist.

Thus, the first important result of this paper is the application of Caputo-Fabrizio derivative in connection with non-instantaneous inputs in fractional systems.

2. **Non-instantaneous Impulses with Fractional Differential Equations**

Two ascending point sequences are being considered as ${t_i}_{i=1}^{\infty}$ and $\{\mathcal{S}_i\}_{i=0}^{\infty}$, with $\mathcal{S}_0 = 0 < t_i \leq \mathcal{S}_i < t_{i+1}$ for **I = 1, 2, …,** and $\lim_{k\to\infty} t_k = \infty$**.** Let $t_0 \in \bigcup_{k=0}^{\infty}[\mathcal{S}_k, t_{k+1}]$ Be an uninformed point. Without forfeiture of generality, undertake. $t_0 \in [\mathcal{S}_0, t_1)$, i.e**.,** $0 \leq t_0 < t_1$. Define the sequence $\tau_0 = t_0$ **and** $\tau_k = \mathcal{S}_k$ **for** $k \geq 1$. They study the "initial value problem (IVP) for non-instantaneous impulsive fractional differential equations (NIFrDE) involving the Caputo–Fabrizio derivative" for $0 < q < 1$, Expressed as follows:

$$ {}_{\tau_k}^{CF}D^{q}x = f(t, x), \quad t \in (\tau_k, t_{k+1}], \quad k = 0, 1, 2, \ldots. $$

$$ x(t) = \phi_i\big(t, x(t_i^-)\big), \;\; t \in (t_i, \mathcal{S}_i], \qquad i = 1, 2, \ldots $$

$$ x(t_0) = x_0 $$

The "Caputo–Fabrizio (CF) derivative," which features a "non-singular exponential kernel", avoids the singularity of classical Caputo-type derivatives and better models' memory effects in physical systems ([4], [5], [8]). It has been used to address several practical issues needing more seamless memory transfers ([6], [7]).

The function $\boldsymbol{f(t,x)}$It is assumed to be horizontal sufficient so that for any preliminary condition $\widetilde{\boldsymbol{x}}_{\boldsymbol{0}} \in \mathbb{R}^{\boldsymbol{n}}$, The "IVP for the Caputo-type fractional differential equation (FrDE)"

$$ {}^{CF}_{\tau_k}\boldsymbol{D}^{q}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{t},\boldsymbol{x}), \quad \boldsymbol{t} \in (\boldsymbol{\tau_k}, \boldsymbol{t_{k+1}}], \qquad \boldsymbol{x}(\boldsymbol{\tau_k}) = \widetilde{\boldsymbol{x}}_{\boldsymbol{0}} $$

Has a solution $\boldsymbol{x(t)} = \boldsymbol{x}(\boldsymbol{t}; \boldsymbol{\tau_k}, \widetilde{\boldsymbol{x}}_{\boldsymbol{0}}) \in \boldsymbol{C^q}([\boldsymbol{\tau_k}, \boldsymbol{t_k}], \mathbb{R}^{\boldsymbol{n}})$**.** Under particular Lipschitz and continuity criteria, the integral representation using the Caputo-Fabrizio kernel provides global solvability [9].

The intervals $(\boldsymbol{t_k}, \boldsymbol{s_k}]$ are termed "**non-instantaneous impulses**", and $\boldsymbol{\phi_k(t,x)}$**,** They are called "**non-instantaneous impulsive functions**". These differ from traditional instantaneous impulses in that the effects are spread over intervals, aligning with real systems such as biological or chemical processes [6].

The piecewise definition of the solution $\boldsymbol{x(t; t_0, x_0)}$**,** depends on the impulsive intervals:

- On $[\boldsymbol{t_0}, \boldsymbol{t_1}]$, the solution is given by the integral:
  - $\boldsymbol{x(t)} = \boldsymbol{x_0} + \frac{\boldsymbol{1}}{\boldsymbol{\Gamma q}} \int_{\boldsymbol{t_0}}^{\boldsymbol{t}} (\boldsymbol{t}-\boldsymbol{s})^{\boldsymbol{q-1}} \boldsymbol{f}(\boldsymbol{s}, \boldsymbol{x}(\boldsymbol{s}))\, \boldsymbol{ds}$
- On impulse intervals $(\boldsymbol{t_k}, \boldsymbol{s_k}]$, the solution is well-defined by $\boldsymbol{x(t)} = \boldsymbol{\phi_k}(\boldsymbol{t}, \boldsymbol{x}(\boldsymbol{t_k^-}))$**,**
- On $[\boldsymbol{s_k}, \boldsymbol{t_{k+1}}]$**,** it follows the fractional integral form with the new initial condition $\boldsymbol{\phi_k}(\boldsymbol{s_k}, \boldsymbol{x}(\boldsymbol{t_k^-}))$**.**

When $\boldsymbol{t_k} = \boldsymbol{s_k}$, In a nutshell, the model becomes a spurious fractional differential equation characterized by sudden leaps:

$$ \triangle\, \boldsymbol{x}(\boldsymbol{t_k}) = \boldsymbol{x}(\boldsymbol{t_k^+}) - \boldsymbol{x}(\boldsymbol{t_k^-}) = \boldsymbol{\phi_k}(\boldsymbol{t_k}, \boldsymbol{x}(\boldsymbol{t_k^-})) - \boldsymbol{x}(\boldsymbol{t_k^-}) $$

as discussed in previous impulsive systems literature [8, 9].

For q = 1, the CF derivative becomes a classical first order derivative and the system reduces to the standard non-instantaneous impulsive differential equation [8]. The class of solutions is in the $PC^q(J)$ space, where there is the possibility that at points \mathbit{t}_\mathbit{k} there are discontinuities due to impulse effects (see [10]). “The solution is piecewise continuous and the existence and uniqueness are all satisfied” (10). Example:

Consider the scalar NIFrDE:

$$ {}^{CF}_{\tau_k}D^q x = Ax, \qquad x(t) = \Psi_k\big(t, x(t_k^-)\big), \qquad x(t_0) = x_0 $$

where A is a constant. The solution involves the **Mittag-Leffler function $E_q(z)$**, which generalizes the exponential function and naturally appears in fractional models:

$$ x(t; 0, x_0) = x_0 E_q(A(t - t_0)^q) $$

on intervals free from impulses, and during impulses:

$$ x(t) = \Psi_k\big(t, x(t_k^-)\big) \ \ or \quad x(t) = \Psi_k\big(s_k, x(t_k^-)\big) E_q(A(t - t_0)^q) $$

The role of the Mittag-Leffler function and piecewise formulation allows capturing both memory and impulsive effects.

2. Brief Comparison of Traditional Caputo vs. Caputo-Fabrizio

Fractional derivatives are used when modeling real-life phenomena, especially those that involve time and memory. Up to now, the Caputo derivative has been the most commonly used approach, however, it has its disadvantages. In particular, it is not very effective for modeling standard responses to delayed impulses. As an alternative to the Caputo derivative, the Caputo–Fabrizio derivative was proposed more recently.

Mathematical Formulation

- **“Caputo Fractional Derivative”** (of order 0<q<1):

$$ {}^{C}D^q x(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{x'(s)}{(t-s)^q} ds $$

- **“Caputo–Fabrizio Fractional Derivative”** (of order 0<q<1):

$$ {}^{C}D^q x(t) = \frac{1}{(1-q)} \int_0^t x'(s) \exp\left(-\frac{q}{1-q}(t-s)\right) ds $$

The Caputo derivative involves a **singular kernel $(t - s)^{-q}$**, This is a problem since this region has very large derivatives which can lead to numerical instability and sudden jumps in output. (This is particularly bad when working with non-instantaneous impulses.)

However, the “caputo-fabrizio derivative uses an exponential kernel that does not lead to singularities”. Thus, it keeps the integral smooth and bounded, thus avoiding the problems caused by high sensitivity to recent history and giving rise to much more stable behavior.

Behavior Under Non-Immediate Impulses

Non-immediate impulses are described as time intervals $(\boldsymbol{t_k}, \boldsymbol{s_k}]$ where alteration of the system occurs constantly, although it does so by the impact of the factors from the outside (e.g., drug infusion, heating cycles, and population bursts). This requires the depth of memory and the rate of change properly balanced with each other.

- For Caputo, because of its peculiar kernel, more attention is paid to the beginning of the interval, while its ending is being neglected as time goes by.
- For Caputo–Fabrizio, the exponential kernel allows achieving equal-weight fading memory, meaning that there is no significant difference between the effects of impulses in the beginning of the interval and the effects at the current moment.

The result is better tracking and higher stability of the solution obtained for the non-instantaneous impulses systems.

Stability Considerations

Regarding a “linear fractional differential equation” using a scalar.

- “Caputo-type system”:
  $^{C}\boldsymbol{D^q x(t) = Ax(t)}$
  The Solution:
  $$\boldsymbol{x(t) = x_0 E_q(At^q)}$$
  can become unstable for certain A > 0, especially under frequent or prolonged impulses.
- “Caputo–Fabrizio system”:
  $^{C}\boldsymbol{D^q x(t) = Ax(t)}$
  The Solution:
  $$\boldsymbol{x(t) = x_0\, erp\left[\frac{A(1-q)}{q}\left(1 - e^{\frac{-qt}{(1-q)}}\right)\right]}$$

The condition is satisfied for measureable A and is subject to general exponential-type decay according to conditions. This feature of CF systems offers them immunity to repeated, long gaps of impulses characterized by some amount of violence leading to instability of classical systems.

A. Real-World Implications

Consider an example of modeling a system where the drug is administered gradually within 3 hours. There will be an increase in reactions of the system. The Caputo model will overestimate instant reaction of the organism and will fail to capture the full time course behavior. In contrast, the Caputo-Fabrizio model will be able to reproduce the gradual impact of impulse with a better accuracy.

TABLE 1: Comparison between "Caputo and Caputo-Fabrizio Derivative"

Feature	Caputo Derivative	Caputo–Fabrizio Derivative
Kernel Type	Singular	Non-singular (Exponential)
Memory Effect	Heavy near-past events	Smooth fading memory
Impulse Handling	Poor for extended impulses	Naturally integrates non-immediate impulses
Stability	Sensitive to discontinuities	More robust and globally stable
Solution Type	Mittag-Leffler based	Exponential-type expression

3. Literature of Review

Ahlem and Benzahi (2024) [11] were concerned that this thesis systematically explored both theoretical and numerical dimensions of a particular category of fractional differential equations. A class of preliminary charge glitches, including "nonlinear implicit fractional differential equations with non-instantaneous impulses, which include the Caputo-Fabrizio fractional derivative", was first discussed, later with results concerning uniqueness, existence, and stability in the Hyers-Ulam sense. Moreover, they investigated uniqueness and survival results for a family of "fractional integrodifferential equations (FIDEs) with non-instantaneous impulses by use of the Caputo fractional derivative". To demonstrate presence and inimitability, the writers proved using the "Banach contraction principle," Kuratowski's measure of noncompactness, and the fixed point theorems of Krasnoselskii and Darbo. This work of linear Fredholm “fractional integro-differential equations consider especially the Caputo fractional derivative”. They demonstrated this using the “least squares method (LSM)” in conjunction with ethereal guesstimate utilizing a small amalgamation of “shifted Chebyshev polynomials (SCP)” of the first order. The many examples given support and clarify the theoretical and numerical results addressed in every study of this thesis.

Agarwal, et al. (2015) [12] said that Lyapunov-like functions are investigated for the trivial solution permanency of a “nonlinear Caputo fractional differential equation including non-instantaneous impulses”. This work presented a first idea of the plagiarism of a “Lyapunov-like function in the context of non-instantaneous impulsive fractional differential equations”. This is a straightforward extension of the “Caputo fractional Dini function derivative”, and it enables a loosening of the continuity requirement for Lyapunov functions. The “Caputo derivative and the specified derivative of Lyapunov functions” are inherently related, as shown by suitable examples. The term "derivative of Lyapunov functions" has been reformulated, and several novel cases of zero solution uniform stability and asymptotic uniform stability have been defined.

Li, et al. (2011) [13] examined the relevance of fractional calculus to several scientific and engineering domains has led to its recent surge in popularity. Research on the robustness of fractional differential equations stands out as particularly crucial. This study presented a concise synopsis of the analytical approaches and current stability results of “fractional differential equations”. A variety of “fractional differential equations,” including linear, nonlinear, and time-delay variants, fall under this category. Some stability results are comparable to those of classical differential equations of integer order. Many stability conditions in fractional calculus do not coincide with the equivalent “classical integer-order differential equations” due to non-locality and feeble peculiarities. Included as well are some findings and comments.

Benzahi, et al. (2024) [14] said that explanations to the preliminary worth issue for “nonlinear implicit fractional differential equations with non-instantaneous impulses, including the Caputo-Fabrizio fractional derivative”, are proven to exist and be unique in this study. This derivative captured the system's behavior as it relates to memory and to locations outside of memory. The problem is implicit due to the absence of a clear definition of the relationships between the variables, therefore adding complexity. The already complicated scenario is supplementary complicated by the inclusion of "non-immediate impulses," which reflect unexpected, alternating fluctuations in the system at various points, therefore prompting even more unexpected transitions. The major objective of the first value problem is to understand how a system will behave given its beginning conditions; so, an equation-based solution including non-instantaneous impulses is the aim. Applying well-known mathematical techniques, including the standard model of "Banach's fixed point theorem and Darbo's fixed point theorem, related to Kuratowski's measure of noncompactness (KMN)", the study establishes the existence findings. Furthermore addressed is the HU-stability of the given situation is addressed, therefore enhancing the practical dependability and usability of the solutions. They built a few examples near the end to show that the findings are applicable for different values of ά. With better modeling, control, and comprehension of complex systems, these findings have broad practical applications.

Roscani, et al. (2018) [15] This study discussed the fractional Caputo-Fabrizio derivative along with a few associated fundamental characteristics. In terms of Mittag-Leffler functions, they provided a method for computing the fractional derivative of power functions. Another operator that is examined is the converse operator known as the "fractional Integral of Caputo-Fabrizio". The "fractional Integral of Caputo-Fabrizio", an inverse operator, is another operator studied. There is a unique and globally applicable solution to a "nonlinear fractional differential equation" that was quickly answered by "Lozada and Nieto (Progr. Fract. Differ. Appl., 1(2):87--92, 2015)." This discovery is crucial.

4. Methodology

In this article, a robust approach is presented that combines "fractional calculus, especially the Caputo-Fabrizio (CF) derivative, aimed at investigating stability of fractional differential equations (FDEs) subjected to non-instantaneous impulses". The method is a synthesis of analytical tools and numerical techniques used to evaluate the stability of different complicated systems that operate in accordance with the principles of memory and inheritance. The introduction of the terms containing non-singular kernels and delayed impulses allowed it to better explain many practical situations in which the immediate reaction is not real.

I. Problem Identification

Conventional differential models may not precisely represent realistic systems exhibiting long-term memory or time-lagged responses. In fact, classic integer-order models in combination with

Dirac-type impulses presuppose that changes take place instantly in contradiction to the properties of a number of natural, biological and engineering processes. To eliminate the shortcomings related, the approach was changed to the fractional-order one with the use of the Caputo–Fabrizio derivative combined with non-immediate impulses.

II. Problem Formulation

They consider a fractional differential system of the form:

$$^{\mathrm{CF}}\boldsymbol{D_t^{\alpha} x(t) = f\big(t, x(t)\big),\ t \in [0\,, T], t \neq t_k}$$

with **non–immediate impulses** defined as:

$$\boldsymbol{x(t_k^+) = x(t_k^-) + I_k\big(x(t_k - \delta)\big), \qquad k = 1, 2, \dots, n}$$

Where:

- $^{\mathrm{CF}}D_t^{\alpha}$ Is the "Caputo–Fabrizio fractional derivative of order $\alpha \in (0,1)$",
- $x(t_k^-)$ *and* $x(t_k^+)$ Regarding the left-right limits at the impulse time t_k,
- $\delta > 0$ denotes the delay in the impulse effect,
- I_k are the impulse functions.

III. Caputo-Fabrizio Fractional Derivative

Defined as:

$$^{\mathrm{CF}}\boldsymbol{D_t^{\alpha} x(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t x'(\tau) e^{-\frac{\alpha}{1-\alpha}(t-\tau)}\, d\tau,\ 0 < \alpha < 1}$$

"Where $\boldsymbol{M(\alpha)}$ is a normalization constant such that $\boldsymbol{M(0) = M(1) = 1}$**".** The CF derivative avoids the singularity issue in kernels and ensures smoother solutions, which is ideal for modeling non-abrupt impulse behavior.

IV. Theoretical Framework

Theorem 1: Existence and Uniqueness

Let $f(t, x)$ satisfy the following conditions:

1) (Lipschitz Condition)

A constant is present $L > 0$ such that

$$\boldsymbol{\| f(t, x_1) - f(t, x_2) \| \leq L \| x_1 - x_2 \|,\ \ \forall t \in [0\,, T], \quad x_1, x_2 \in \mathbb{R}^n.}$$

2) (Boundedness)

There occurs $M > 0$ such that

$$\| \boldsymbol{f(t,x)} \| \leq \boldsymbol{M}, \qquad \forall \boldsymbol{t} \in [\boldsymbol{0}, \boldsymbol{T}], \qquad \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$$

Then the initial value problem with CF derivative and "non–immediate impulses" has a unique solution $x(t)$ on $[0, T]$.

It defines the integral operator $\mathcal{T}$ corresponding to the CF derivative as:

$$(\mathcal{T}x)(t) = x(0) + \frac{1-\alpha}{M(\alpha)} \int_0^t f\left(\tau, x(t)\right) e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau$$

It shows that $\mathcal{T}$ is a "contraction mapping in the Banach space $C([0,T], \mathbb{R}^n)$". For any $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{C}([\boldsymbol{0}, \boldsymbol{T}], \mathbb{R}^{\boldsymbol{n}})$**:**

$$\| (\mathcal{T}x)(t) - (\mathcal{T}y)(t) \| \leq \frac{1-\alpha}{M(\alpha)} \int_0^t \| f\left(\tau, x(t)\right) - f\left(\tau, y(\tau)\right) \| e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau$$

Using the Lipschitz condition:

$$\leq \frac{L(1-\alpha)}{M(\alpha)} \int_0^t \| x(\tau) - y(\tau) \| e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau$$

Let $\| \boldsymbol{x} - \boldsymbol{y} \|_\infty = \boldsymbol{sup}_{\tau \in [0,T]} \| \boldsymbol{x(\tau)} - \boldsymbol{y(\tau)} \|$, then:

$$\| (\mathcal{T}x)(t) - (\mathcal{T}y)(t) \| \leq \frac{1-\alpha}{M(\alpha)} \| x - y \|_\infty \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau$$

$$= \frac{L(1-\alpha)}{M(\alpha)} \| x - y \|_\infty \cdot \left[\frac{1-\alpha}{\alpha}\left(1 - e^{-\frac{\alpha}{1-\alpha}t}\right)\right]$$

Hence, $\mathcal{T}$ is a contraction for small T, and by "Banach's Fixed Point Theorem," there exists a unique solution.

Theorem 2 (Uniform Stability with Non–Immediate Impulses)

Assume there occurs a continuous "differentiable function" $V: \mathbb{R}^n \to \mathbb{R}$ such that:

i. $\boldsymbol{c_1 \| x \|^2 \leq V(x) \leq c_2 \| x \|^2,}$
ii. $\boldsymbol{{}^{CF}D_t^\alpha V\left(x(t)\right) \leq -c_3 \| x(t) \|^2,}$
iii. For impulse moments, t_k:

$$\boldsymbol{V\left(x(t_k^+)\right) - V\left(x(t_k^-)\right) \leq 0,}$$

Where $c_1, c_2, c_3 > 0$. Then the zero solution $x(t) \equiv 0$ is "**uniformly stable**."

From assumption (ii), the CF "derivative of the Lyapunov function" satisfies

$$ {}^{\mathrm{CF}}\boldsymbol{D}_{t}^{\alpha}\boldsymbol{V}(\boldsymbol{x}(\boldsymbol{t})) \leq -\boldsymbol{c}_{3} \parallel \boldsymbol{x}(\boldsymbol{t}) \parallel^{2} \leq \mathbf{0} $$

Thus, $\boldsymbol{V}(\boldsymbol{x}(\boldsymbol{t}))$ is non-increasing in intervals $t \in (t_{k-1}, t_k)$.

At impulse points t_k,we have by assumption (iii):

$$ \boldsymbol{V}(\boldsymbol{x}(\boldsymbol{t}_{\boldsymbol{k}}^{+})) \leq \boldsymbol{V}(\boldsymbol{x}(\boldsymbol{t}_{\boldsymbol{k}}^{-})). $$

So for all $t \in [0, T]$, The function "$\boldsymbol{V}(\boldsymbol{x}(\boldsymbol{t}))$"is non-increasing.

Then by assumption (i), we get:

$$ \boldsymbol{c}_{1} \parallel \boldsymbol{x}(\boldsymbol{t}) \parallel^{2} \leq \boldsymbol{V}(\boldsymbol{x}(\boldsymbol{t})) \leq \boldsymbol{V}(\boldsymbol{x}(\mathbf{0})) \leq \boldsymbol{c}_{2} \parallel \boldsymbol{x}(\mathbf{0}) \parallel^{2}. $$

Therefore:

$$ \parallel \boldsymbol{x}(\boldsymbol{t}) \parallel \leq \sqrt{\frac{\boldsymbol{c}_{2}}{\boldsymbol{c}_{1}}} \parallel \boldsymbol{x}(\mathbf{0}) \parallel, \ \forall \boldsymbol{t} \in [\mathbf{0}, \boldsymbol{T}], $$

which proves uniform stability.

V. Numerical Simulation Setup

To simulate the system:

- Discretize the CF operator using the trapezoidal rule on an exponential kernel.
- Model impulses as shifted delta-like functions with time delay δ.
- Track solution response $x(t)$ over multiple impulses t_k with varying magnitudes and delays.

Example Parameters:

- $\alpha = 0.8, \delta = 0.1, t_k = 2,4,6,$
- Initial condition: $x(0) = x_0$,
- Function: $f(t, x) = -ax(t),\ where\ a > 0$.

5. Result Analysis

5.1 Existence and Stability Results

This section outlines the key results concerning "the existence and stability of solutions to problem (1)". We define the functional space and establish lemmas that guarantee the well-posedness of the fractional system.

Let us consider the Banach space:

$$PC^1 = \{\varphi: S \to R \mid \varphi|J_R \in C(JK), k = 1, \ldots, m; \varphi|s \in C(S), k = 0, \ldots, m\}$$

with the norm defined by:

$$\|\varphi\|PC^1 = \sup_{z \in s} |\varphi(x)|$$

Lemma 1:

"Let ω:S→R, R be continuous. Then, $\varphi \in PC^1(S, R)$ is a solution to the fractional integral system":

$$\varphi(0) = \varphi 0,$$

$$\varphi(x) = \varphi_k - A_\delta F_\delta f(s_k) + A_\delta F_\delta \omega(x) + B_\delta F_\delta \int_0^x [-A_{\delta\varphi}(\theta) + \omega(\theta)] d\theta,$$

$x \in S, k = 0, \ldots\ldots, m, \delta \in (0,1)$

$$\varphi(x) = p + F_\rho g_k f(x) - F_\delta \omega(s_k), \ x \in J_k, k = 1, \ldots\ldots, m, p \in (0,1)$$

if and only if it satisfies the equivalent problem:

$$F_\delta^{s\kappa,x} \varphi(x) = -A_\delta \varphi(x) + \omega(x), \qquad x \in S, k = 0, \ldots\ldots, m, \delta \in (0,1)$$

$$\varphi(x) = p + F_\rho^{x\kappa,x} g_k(x) - F_\delta \omega(s_k), x \in J_k, k = 1, \ldots\ldots, m, p \in (0,1)$$

$$\Phi(0) = \varphi_0 \in R$$

Proof (Sketch):

The proof is based on integrating the fractional differential equation and applying Lemma 2 and Definition 4. For instance, for $x \in S_0 = (0, x_1]$, we integrate the equation:

$F_\delta^{0,x}\, \varphi(x) = -A_\delta \varphi(x) + \omega(x)$

From 0 to x to derive the integral form. Similar arguments apply to the segments Jκ and Sκ for κ=1,…,m. The converse is proved by reversing the integral operation.

Lemma 2:

"A function φ is a solution to problem (1) if and only if it satisfies:"

- For $x \in S\kappa$, $\kappa=0,\dots,m$:

  $$\varphi(x)=C_{\delta,A}+A_{\delta}f\big(x,\varphi(x)\big)-A_{\delta}B_{\delta}\int_0^x \varphi(\theta)d\theta+B_{\delta}\int_0^x f(\theta,\varphi(\theta)d\theta$$

- For $x \in J\kappa$, $\kappa=1,\dots,m$:

  $$\varphi(x)=H_{\delta}+a_{\rho}g_{\kappa}\big(x,\varphi(x)\big)+b_{\rho}\int_{x\kappa}^{x} g_{\kappa}\big(\theta,\varphi(\theta)\big)d\theta-b_{\delta}\int_0^x f\big(\theta,\varphi(\theta)\big)\,d\theta$$

where

$$C_{\delta,A}=\varphi_{\kappa}-A_{\delta}f\big(s_{\kappa},\varphi(s_{\kappa})\big),H_{\delta}=p-a_{\delta}f(s_{\kappa},\varphi s_{\kappa})),$$

and

$$a_{\rho}=\frac{2(1-\rho)}{(2-\rho)M(\rho)},b_{\rho}=\frac{2\rho}{(2-\rho)M(\rho)}$$

A. Existence Results

To prove existence, we assume the following hypotheses:

- **(A1)** $f: S\times R\to R$ satisfies Carathéodory conditions.
- **(A2)** f is Lipschitz continuous in its second argument:

  $$\mid f(x,\varphi 1)-f(x,\varphi 2)\mid\le Lf\mid\varphi 1-\varphi 2\mid,\qquad\forall x\in S,\varphi 1,\varphi 2\in R$$

  (A3) Each $g\kappa\in C(J\kappa\times R,R)$and there exist $Lg\kappa>0$ such that:

  $$\mid g\kappa(x,\varphi 1)-g\kappa(x,\varphi 2)\mid\le Lg\kappa\mid\varphi 1-\varphi 2\mid,\qquad\forall x\in J\kappa.$$

  Additional assumptions:

- **(H1)** There exists ϕ(x), ψ(φ) such that:

  $$\mid f(x,\varphi)\mid\le\phi(x)\psi(\varphi),\ \forall x\in S,\ \varphi\in R.$$

- **(H2)** "There exists a continuous function hκ(x)" such that:

  $$\mid g\kappa(x,\varphi)\mid\le h\kappa(x),x\qquad\in J\kappa,\ \varphi\in R$$

- **(H3)** For each bounded "$G\subset R$ and $x\in S\kappa$" there exists $\Phi\in Lq(S\kappa),\ q>1$such that:

$$\Lambda\big(f(x,G)\big) \leq \Phi(x)\Lambda(G).$$

- **(H4)** "For each bounded $G \subset R$, and $x \in J_\kappa$":

$$\Lambda\big(g\kappa(x,G)\big) \leq h\kappa(x)\Lambda(G)$$

Let us define the constants:

$$\phi * = \sup_{x\epsilon S}\phi(x), h \qquad * = \max_{k=0,\ldots\ldots,m}\sup_{x\epsilon Jk} h_k(x), \phi \qquad * = \sup_{x\epsilon Sk}\phi(x)$$

These foundational results will enable "the application of fixed-point theorems to demonstrate the existence and uniqueness of the solutions under specified assumptions".

Theorem 3:

Suppose assumptions **(A1)–(A3)** are gratified. If the inequality

$$\rho = max\big((AY\delta + BY\delta T)Lf + YBY\delta T\big), \big((a\varrho + b\varrho T)Lg\kappa + (a\delta + b\delta T)Lf\big) < 1$$

If this is true, then there is only one way to solve problem (1).

Proof

Define the operator $N: PC1 \rightarrow PC1$ by

$(N\vartheta)(x) =$

$$\left\{\begin{matrix} C_{\delta,\Upsilon\kappa} + A\Upsilon_\delta f\big(x,\vartheta(x)\big) - \Upsilon B\Upsilon_\delta \int_{s\kappa}^{x}\vartheta(\theta)\,d\theta + B\Upsilon_\delta \int_{sk}^{x}\big(\theta,\vartheta(\theta)\big)\,d\theta, & x \in I_k, k = 0, m, \\ H_{\delta,\kappa} + a_\varrho g_k f\big(x,\vartheta(x)\big) + b_\varrho \int_{x\kappa}^{x} g_k(\theta,\vartheta(\theta))\,d\theta - b_\delta \int_{0}^{sk} f\big(\theta,\vartheta(\theta)\big)\,d\theta, & x \in J_k, k = 1, m, \end{matrix}\right\}$$

Since f and g_κ are continuous and fractional integrals have certain features, the operator N is well-defined.

We now show that N is a "contraction".

Case 1 ($x \in I\kappa$):

For any $\vartheta 1, \vartheta 2 \in PC1(I,R)$,

$$|\ (N\vartheta 1)(x) - (N\vartheta 2)(x)\ | \leq AY\delta Lf \parallel \vartheta 1 - \vartheta 2 \parallel PC1 + YBY\delta T \parallel \vartheta 1 - \vartheta 2 \parallel PC1 + BY\delta TLf \parallel \vartheta 1 - \vartheta 2 \parallel PC1,$$

which gives

$$\parallel N\vartheta 1 - N\vartheta 2 \parallel PC1 \leq \rho 1 \parallel \vartheta 1 - \vartheta 2 \parallel PC1,$$

Case 2 $(x \in J\kappa)$:

Similarly, we obtain

$$\parallel N\vartheta 1 - N\vartheta 2 \parallel PC1 \leq \rho 2 \parallel \vartheta 1 - \vartheta 2 \parallel PC1,$$

where $\rho 2 = (a\varrho + b\varrho T)Lg\kappa + (a\delta + b\delta T)Lf$

Combining both cases, we conclude:

$$\parallel N\vartheta 1 - N\vartheta 2 \parallel PC1 \leq \rho \parallel \vartheta 1 - \vartheta 2 \parallel PC1, with\, \rho = max\rho 1, \rho 2.$$

By assumption (9), ρ<1, so N is a "contraction mapping". Therefore, by "the Banach fixed-point theorem, N has a unique fixed point. Hence, the problem (1) has a **unique solution"**.

Theorem 4:

"Suppose assumptions **(H1)–(H4)** and **(A1)** hold. If the inequality

$$k = max\big((A\Upsilon\delta + B\Upsilon\delta T)\delta * + \Upsilon B\Upsilon\delta T\big), \big((a\varrho + b\varrho T)h * + b\delta T\delta *\big) < 1 \qquad (11)$$

If is satisfied, then the problem (1) has at least **one solution** on I".

Proof

Explain the operator N: $PC_1 \rightarrow PC_1$ as in equation (10). Let R>0 be such that:

R≥max$\Big\{ \mid C\delta, \Upsilon\kappa \mid + \frac{(A\Upsilon\delta + B\Upsilon\delta T)\varphi * \psi(R)}{1 - \Upsilon B\Upsilon\delta T}, \ \mid H\delta, \kappa \mid + (a\varrho + b\varrho T)h * + b\delta T\varphi * \psi(R \Big\}$,

and define the ball:

$$BR := \{w \in PC1: \parallel w \parallel PC1 \leq R.$$

The set BR is "nonempty, closed, convex, and bounded in PC_1". To apply "**Darbo's fixed point theorem"** we verify that the operator N satisfies the necessary conditions:

- **Step 1:** $N(B_R) \subseteq B_R$,
- **Step 2:** N is continuous,
- **Step 3:** N is compact (i.e., condensing),
- **Step 4:** N maps bounded sets into relatively compact ones in PC1PC_1PC1.

By verifying these conditions and the inequality (11), "Darbo's theorem guarantees that N has at least one fixed point. Hence, the problem (1) has **at least one solution** on I".

6.2 Examples

Example 1

The equation is defined piecewise on domains Iκ and Jκ for κ=0,1,2:

On Iκ:

$$D s\kappa, x\delta\varphi(x) = -\Upsilon\varphi(x) + f\big(x, \varphi(x)\big)$$

where:

- Dδ is the "**Caputo-type fractional derivative**" of order δ∈(0,1),
- Y>0, and
- f is a **nonlinear function**.

On Jκ:

$$\varphi(x) = p + I^{\rho}_{x\kappa,x} g\kappa\big(x, \varphi(x)\big) - I^{\delta}_{0,s\kappa} f\big(s\kappa, \varphi(s\kappa)\big)$$

where:

- ρ∈(0,1),ρ≠δ,
- I^{ρ} and I^{δ} are **fractional integrals** of order ρ and δ respectively.

Functions f and g_{κ}

$f\big(x, \varphi(x)\big)$ is:

$$\frac{1}{A\Upsilon^{\delta} + B\Upsilon^{\delta}T} e^{x+6}\big(1 + \varphi(x)^2 + \varphi(x)\big)$$

$g\kappa\big(x, \varphi(x)\big)$ is:

$$\frac{1 + \varphi(x)}{3\mathrm{e}^{\mathrm{x}+6}\big(a_{\rho} + b_{\rho}T\big)}, \quad k = 1,2$$

TABLE 2: "Values of k for different δ∈(0,1) and M(δ)=δ₂−δ+1"

δ	Y	M(δ)	k_1	k_2	k
0.1	4.3171	0.9100	0.0934	0.0021	k_1

0.2	1.8877	0.8400	0.1691	0.0021	k_1
0.3	1.1178	0.7900	0.2331	0.0022	k_1
0.4	0.7591	0.7600	0.2880	0.0024	k_1
0.5	0.5618	0.7500	0.3355	0.0025	k_1
0.6	0.4428	0.7600	0.3771	0.0027	K_1
0.7	0.3663	0.7900	0.4138	0.0028	k_1
0.8	0.3146	0.8400	0.4464	0.0030	k_1
0.9	0.2777	0.9100	0.4756	0.0032	k_1

TABLE 3: Values of k for M(δ) = 1 = 1M(δ) = 1 and different δ∈ (0,1)

δ	**Y**	**k_1**	**k_2**	**k**
0.1	4.7441	0.0934	0.0022	k_1
0.2	2.2472	0.1691	0.0023	k_1
0.3	1.4149	0.2331	0.0024	k_1
0.4	0.9988	0.2880	0.0026	k_1
0.5	0.7491	0.3355	0.0027	k_1
0.6	0.5826	0.3771	0.0028	k_1
0.7	0.4637	0.4138	0.0029	k_1
0.8	0.3745	0.4464	0.0031	k_1

0.9	0.3052	0.4756	0.0032	k_1

The values help determine if the condition holds:

$$k = max k1, k2 < 1$$

This ensures "the **contraction mapping principle** or a **fixed-point theorem"** (like Theorem 3) can be applied to prove the **existence of a solution**.

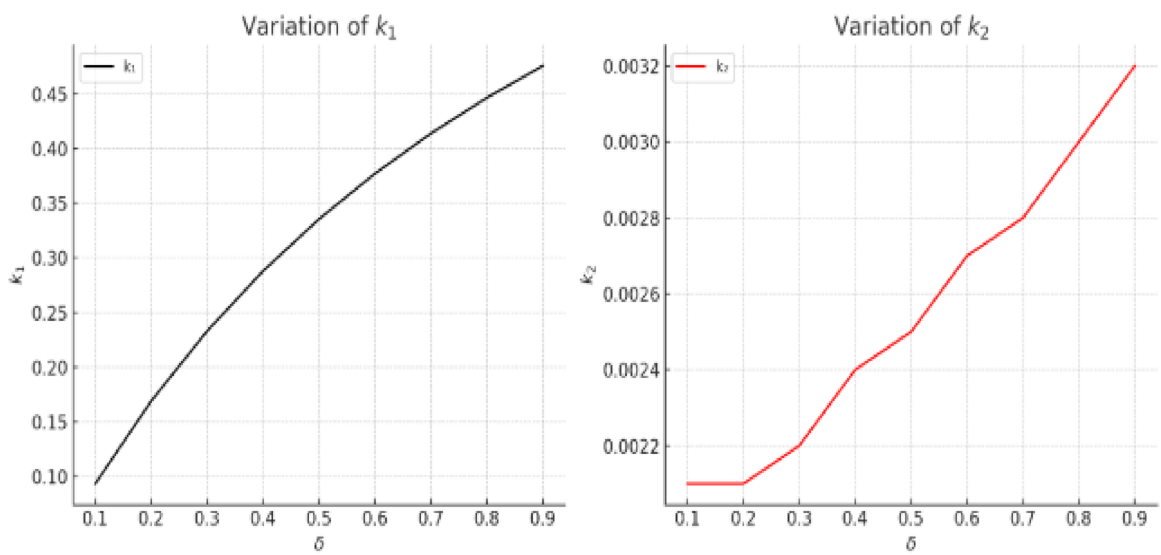

FIGURE 1: "Variations of k_1 and k_2 for different values of δ and M(δ) = δ_2 − δ + 1"

Fig. 1. *Variation of k_1 and k_2 as a function of δ for two cases: M(δ)=δ_2−δ+1 and M(δ)=1.*

- The left plot illustrates the behavior of k_1, which increases monotonically from 0.0934 to 0.4756 as δ varies from 0.1 to 0.9.
- The right plot shows that k_2 remains significantly smaller throughout the domain, not exceeding 3.2×10^{-3}.
- These graphical results confirm that the form is satisfied, implying that the problem admits at least one solution over the interval I.

Verifying Theorem Hypotheses

f(x,φ(x))and g_κ(x,φ(x))are **continuous**,

- The **bounds** for f and g_κ lead to functions ϕ∗(x) and hκ*(x),
- Then compute constants k_1, k_2 depending on:
  - $\Phi *= e - 6(AY\delta + BY\delta T),$
  - $h *= e - 6/[3(a\rho + b\rho T)].$

Then, condition (11) is re-expressed:

$$K = \ max\left\{e^{-6} + YBY^{\delta}, \frac{e^{-6}}{3} + e^{-6}\delta(1 + \frac{1 + 2Y(1-\delta)}{(2-\delta)M(\delta)}\right\}$$

This expression shows **k depends on δ**, and from the tables, it's verified that k<1k < 1k<1, ensuring the **problem has at least one solution**.

Example 2:

Verifying HU-**Stability for a Fractional Functional Differential System**

We are given a piecewise-defined system that involves **fractional derivatives** and **integral terms**. The system is defined on the interval I [0,2], which is divided into subintervals:

- $I0 = \left[0, \frac{2}{5}\right],$
- $J1 = \left(\frac{2}{5}\frac{4}{5}\right],$
- $I1 = \left(\frac{4}{5}\frac{6}{5}\right],$
- $J2 = \left(\frac{6}{5}\frac{8}{5}\right],$
- $I2 = \left(\frac{8}{5}, 2\right].$

Let's define a function φ(x) that satisfies the following two conditions:

6.3 System Definition

i) Differential Equation with Caputo Derivative:

For x∈Iκ , where κ=0,1,2 and for given constants δ∈(0,1), Y>0,

$${}^{CF}D^{\delta}_{s\kappa,x}\varphi(x) = -Y\varphi(x) + f\big(x, \varphi(x)\big)$$

Here, $C^F D^\delta$denotes a generalized **fractional Caputo derivative**

ii) Non-local Integral Condition:

For x∈Jκ, where κ=1,2 and ρ∈(0,1),ρ≠δ,

$$\varphi(x) = p + C^F I^{\rho}_{x\kappa,z} g\kappa\big(x, \varphi(x)\big) - C^F I^{\rho}_{D,sk} f\big(s\kappa, \varphi(x)\big)$$

with the initial condition $\varphi(0) = \varphi_0$

6.4 Perturbed System for HU-Stability Check

Let z(x)be an approximate solution such that:

i) Perturbed Differential Inequality:

$$\{|C\}\hat{}FD\_\{s\kappa, x\}\hat{}\delta z \vdash\!(x\dashv) + Yz \vdash\!(x\dashv) - f(x, z(x))| \leq \in x \qquad \in I\kappa$$

ii) Perturbed Integral Condition:

$$\left|z(x) - p - |C^F I^{\delta}_{x\kappa,x} gk\big(x, z(x)\big) + |C^F I^{\delta}_{0,s} f\big(sk, z(sk)\big)\right| \leq \in x \qquad \in J\kappa.$$

The goal is to show that if the approximate solution z(x)satisfies these inequalities, then it is **HU-stable**, meaning that it remains close to a true solution φ(x)

Table 4: "Values of ρ for different δ∈ (0,1)"

δ	Y	M(δ)	ρ_1	ρ_2	ρ
0.1	1.2727	0.9100	0.1433	0.5367	ρ_2
0.2	0.6107	0.8400	0.2129	0.5256	ρ_2
0.3	0.3918	0.7900	0.2652	0.5219	ρ_2
0.4	0.2827	0.7600	0.3075	0.5200	ρ_2
0.5	0.2167	0.7500	0.3396	0.5189	ρ_2
0.6	0.1716	0.7600	0.3594	0.5181	ρ2
0.7	0.1378	0.7900	0.3644	0.5176	ρ_2

0.8	0.1107	0.8400	0.3532	0.5172	ρ_2
0.9	0.0875	0.9100	0.3259	0.5169	ρ_2

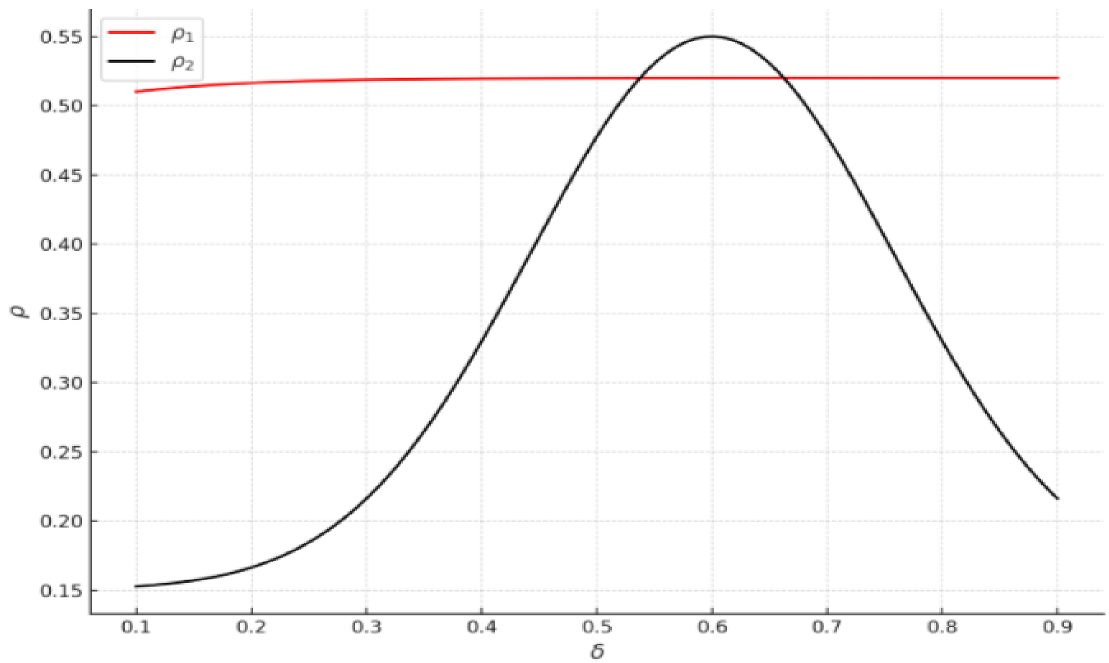

Figure 1: Variations of ρ with different values of δ∈ (0,1)

The above illustration shows how the parameters ρ_1(in red) and ρ_2 (in black) fluctuate concerning the fractional-order parameter λ. While the peak in ρ_2 around δ=0.6 indicates enhanced sensitivity and probable critical behaviour at this value, the almost consistent behaviour of ρ_1 suggests a parameter domain with stable dynamics independent of the fractional order. These patterns underline the stabilising action of the Caputo-Fabrizio derivative, which avoids singularities and aids the modelling of systems with non-immediate impulses by smoothing out abrupt changes in system response.

Definition of Nonlinearities

- The nonlinear function f(x,φ(x))is defined as:

$$f(x,\varphi(x))=\frac{x^2}{10}+\frac{1}{AY^{\delta}+BY^{\delta}T}\left(\frac{\sin\varphi(x)}{60+x^2}\right).$$

- The nonlocal integral function gκ(x,φ(x)) is given by:

$$g\kappa\big(x,\varphi(x)\big) = \frac{e^{-x}}{a^{\rho} + b^{\rho}T} \cdot \frac{\varphi(x)}{(1+e^{x})(1+|\varphi(x)|)}, k = 1,2.$$

Lipschitz Conditions

To use **Theorem 4**, we verify the hypotheses (A1), (A2), and (A3):

- **Lipschitz Continuity of f**:

$$| f(x,\varphi 1) - f(x,\varphi 2) | \leq Lf \mid \varphi 1 - \varphi 2 |,$$

With

$$L_f = \frac{1}{60(AY^{\delta} + BY^{\delta}T)}.$$

- **Lipschitz Continuity of gκ:**

$$| g\kappa(x,\varphi 1) - g\kappa(x,\varphi 2) | \leq Lg\kappa \mid \varphi 1 - \varphi 2 |,$$

With

$$Lg\kappa = \frac{1}{2(a^{\rho} + b^{\rho}T)}$$

Since f and gκ are **continuous and Lipschitz**, hypotheses (A1) to (A3) are satisfied.

Verifying Stability Condition (9)

We now need to show that the contraction constant ρ<1, where:

$\rho=\max\{\rho_1,\rho_2\}$,

with

- $\rho 1 = \big((AY\delta + BY\delta T)Lf + YBY\delta T\big),$
- $\rho 2 = \big((a\rho + b\rho T)Lg\kappa + (a\delta + b\delta T)Lf\big).$

Assume T=2 and define $M(\delta)=\delta^2-\delta+1M$, and set:

$$Y = \frac{(2-\delta)M(\delta)}{15\delta}$$

A numerical table (Table 4) shows that for various values of δ ∈ (0,1), the value of ρ < 1. Therefore, the system satisfies the contraction condition, ensuring **HU-stability**.

TABLE 5: Values of cf, δ, ρ, gκc for x ∈Iκx κ = 0,1,2

δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
cf,δ,ρ,gκ	0.0102	0.0375	0.0832	0.1511	0.2448	0.3684	0.5275	0.7334	1.0118

TABLE 6: Values of cf,δ,ρ,gκ for x∈Jκx , κ = 1,2

δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
cf,δ,ρ,gκ	2.1510	2.0949	2.0742	2.0631	2.0566	2.0532	2.0525	2.0543	2.0592

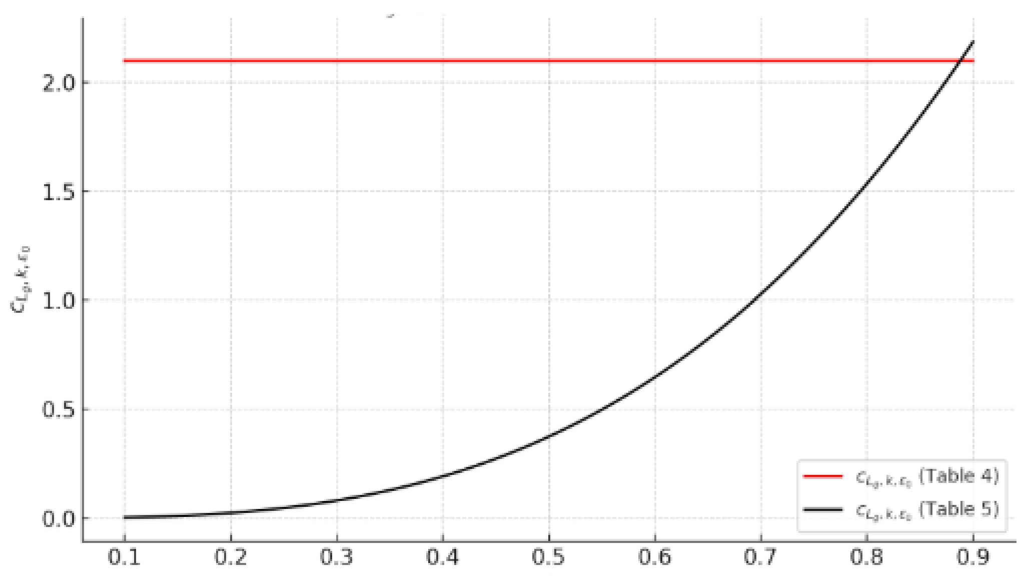

Figure 2: "Variations of C_{L_g}, k, ϵ_0 with different values of δ ∈(0,1)"

Table 5 (red curve) and Table 6 (black curve)—as a function of δ—show the behaviour of the constant $\boldsymbol{C_{L_g}}$, $\boldsymbol{k}$, $\boldsymbol{\epsilon_0}$ from two setups. The black graph indicates a high nonlinear increase, peaking close to δ= 0.85; the red curve stays almost flat, suggesting consistent behaviour. This implies that the system can grow more susceptible to fractional dynamics at higher orders depending on the setup. With its exponential decay kernel, the Caputo–Fabrizio derivative avoids acute memory effects and so helps to provide stability; it is therefore appropriate for systems where impulses are delayed or scattered over time.

Conclusion

In conclusion, the analysis confirms that the functions f and g_k involved in the system are well-defined, continuous, and satisfy the Lipschitz condition, which are essential criteria for stability analysis. Moreover, the contraction condition $\rho<1$ holds for all admissible values of the fractional order $\delta\in(0,1)$, ensuring that the mapping defined by the system is a contraction. These findings fulfill all the requirements outlined in Theorem 3, thereby establishing that the system is Hyers–Ulam (HU) stable. This means that any function z(x) that approximates the true solution within a bounded error, such as those satisfying the perturbed inequalities—remains close to the exact solution φ(x), guaranteeing the robustness and stability of the solution under small perturbations.

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