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Fractional Differential Equations and Their Applications

Anil Kumar*

Department of Mathematics, Government Degree College, Budaun 243601, Uttar Pradesh, India
(affiliated with Mahatma Jyotiba Phule Rohilkhand University, Bareilly, India)
****Corresponding author E-mail:*** - anilk1462@gmail.com

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Abstract

Fractional differential equations (FDEs) are a more advanced version of classical equations that consist of non-integer order derivatives. This modification of differential equations allows modeling complex systems with memory effects, hereditary properties, and anomalous behavior, which cannot be described using classical integer-order models. Over the last few years, fractional differential equations have found broad applications in various areas of science and engineering including physics, biological research, and finance. FDEs are employed in modeling unusual diffusion behavior in physics such as subdiffusion and superdiffusion in nonhomogeneous materials. In biology, FDEs are used to describe memory-based processes including the dynamics of brain networks and viscoelastic materials. In finance, fractional mathematical models help to explain certain aspects of financial market volatility and temporal dependence in financial time series. The field of fractional calculus has undergone extensive evolution, resulting in numerous definitions of fractional derivatives, including the classical Riemann-Liouville derivative, the modern Caputo derivative, and the more advanced Grünwald-Letnikov derivative. The aim of this paper is to succinctly demonstrate basic properties of finite difference equations (FDEs), introduce representative numerical and analytical methods used to resolve these equations, and comment on the wide applicability of these equations in various areas of science and technology. The advantages of the application of finite difference equations in modeling the processes in the real world, the problems of their computational realization, and the opportunities for their implementation in the course of solving scientific problems will be discussed.

Keywords: *Fractional differential equations; Fractional calculus; Caputo derivative; Riemann–Liouville derivative; Grünwald–Letnikov derivative; Memory effects; Anomalous diffusion; Numerical methods; Mathematical modeling; Applications in science and engineering.*

Introduction

The established framework of differential equations has been adapted to account for derivatives of non-integer orders through the introduction of fractional differential equations (FDEs), a revolutionary development in the domains of mathematics and applied science. The application of FDEs contributes to formulating complex systems in a more precise way, such as those with memory, long-range interaction, and abnormal dynamics properties. The idea of fractional calculus existed since the second half of the 17th century, but recently it became popular because of its potential to model phenomena that are not described correctly by classical models of integer order. There have been different applications of fractional differential equations (FDEs) in various industries such as physics, biology, engineering, and finance, among others. FDEs have proved beneficial for modeling processes which are fundamentally nonlocal and time-dependent. Engineering has been changed by the introduction of fractional controllers that outperform conventional controllers in better controlling systems. Fractional models also find application in economics, especially in gaining insight into the dynamics of the market and its volatility. The models are able to reflect the long-run correlations typical of financial time series. The current study illustrates the transformational role of fractional differential equations in understanding complex systems in many areas in the sense that theory and practice are intertwined.

Objectives

1. To Covers the discussion of the nature of the solutions to fractional partial differential equations by means of the mathematical program Mathematica..

Fractional order differential equations of the higher order

In general, the fractional ordinary differential equations can be expressed as follows:

$$F\left(x, y(x), D^{\alpha_1} y(x), D^{\alpha_2} y(x), \cdots, D^{\alpha_n} y(x)\right) = g(x) \qquad (1)$$

where $F(x, y_1, y_2, \cdots, y_n)$ and $g(x)$ who are assigned a function and D^{α_n} the operators of fractional differentiation with real numbers are defined as $\alpha_n > 0$ or complex α_n, $Re\alpha_n > 0$, $n = 1, 2, 3, \cdots, k$ nonlinear differential for the purpose of

$$D^{\alpha} y(x) = f\left(x, y(x)\right) \qquad (2)$$

With real $\alpha > 0$ or complex α $(Re(\alpha) > 0)$, and the linear differential equations

$$\sum_{n=1}^{k} c_n(x) D^{\alpha_n} y(x) + c_0(x) y(x) = f(x) \quad (3)$$

There are a few distinct ways that the fractional differentiation operators (1) and (3) can be expressed. On the other hand, the equations that contain the Riemann Liouville fractional derivative are the ones that are being investigated. In the case of complicated $\alpha \in \mathbb{C}, Re(\alpha) > 0$, When a fractional derivative is defined in this manner,

$$\left(D_{a+}^{\alpha} y\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(I_{a+}^{n-\alpha} y\right)(x),\ n = \left[Re(\alpha)\right] + 1, \quad (4)$$

where $\left(I_{a+}^{n-\alpha} y\right)(x)$ is the fractional integral of order α .

$$\left(I_{a+}^{n-\alpha} y\right)(x) = \frac{1}{\Gamma(x)} \int_a^x \frac{y(t)\mathrm{d}t}{(x-t)^{1-\alpha}} \quad \left(\alpha \in \mathbb{C}, Re(\alpha) > 0\right) \quad (5)$$

$\Gamma(x)$ represents the Gamma function. It should be brought to your attention that the Riemann Liouville method (5) to the definition of fractional integration is an extension of the integration operator $\int_{a+}^{x}$ applied n times:

$$\int_a^x \mathrm{d}t \int_a^t \mathrm{d}t_1 \cdots \int_a^{t_{n-2}} y(t_{n-1}) \mathrm{d}t_{n-1} = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} y(t) \mathrm{d}t \quad (6)$$

(If we apply the formula, then $(n-1)! = \Gamma(n)$) n was substituted for by $\alpha \in \mathbb{C}$ $(Re(\alpha) > 0)$, Furthermore, the fractional differentiation operator and the result of (6) are both (5) D_{a+}^{α} is the opposite of the fractional integration that comes from the center of the left:

$$\left(D_{a+}^{\alpha} I_{a+}^{\alpha}\right)(x) = y(x) \quad \left(\alpha \in \mathbb{C}, Re(\alpha) > 0\right) \quad (7)$$

In order to get the desired effect $y(x)$ from (5). For real $\alpha > 0$, The version of equation (4) is as follows:

$$\left(D_{a+}^{\alpha} I_{a+}^{\alpha}\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y(t)\mathrm{d}t}{(x-t)^{\alpha-n+1}},\ n = [\alpha] + 1, \quad (8)$$

where is the $[\alpha]$ refers to the essential component of α in particular if $0 < \alpha < 1$,

$$\left(D_{a+}^{\alpha} y\right)(x) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{y(t)\mathrm{d}t}{(x-t)^{\alpha}}, \quad (9)$$

And if $\alpha = n \in \mathbb{N} = \{1, 2, \cdots\}$, then $\left(D_{a+}^{\alpha} y\right)(x) = \left(D^n y\right)(x)$, $\left(D = \frac{\mathrm{d}}{\mathrm{d}x}\right)$ consists of the typical derivative of order n.

Applications of Fractional Differential Equations, Fractional Calculus, and Related Topics

There are many complex mathematical ideas that started out as simple ones. One example is the inclusion of the natural number's extension to the real number in certain mathematical computations. Allow me to illustrate it with an example: The factorial of a non-negative integer n is produced by adding all positive integers that are less than or equal to n, denoted by n!. A concept called the Gama function is defined below, though.

$$\Gamma(x)=\int_0^{\infty} t^{x-1}\mathrm{e}^{-t}\mathrm{d}t \qquad (10)$$

The function's one quality that applies to $n\in\mathbb{R}^{+}$ is

$$\Gamma(n+1)=n\Gamma(n) \qquad (11)$$

Therefore, for integers, this function is the same as the factorial. This suggests that the gamma function is just the factorial function extended to real values. To illustrate the formalism mentioned before, the following procedures may be used to construct a factorial of half:

$$\left(\frac{1}{2}\right)!=\Gamma\left(\frac{3}{2}\right)=\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)=\frac{3\sqrt{\pi}}{2} \qquad (12)$$

In Wikipedia's view, the gamma function provides a way to solve the problem of simultaneously connecting all of the points on a smooth curve (x,y)

In Wikipedia's view, the gamma function provides a way to solve the problem of simultaneously connecting all of the points on a smooth curve (x,y) given by $y=(x-1)!$ where x is represented by a positive integer number.

Lemma 1. Let 1 2 < ≤ α . According to the following connections,:

1) if $y\in L^{\infty}(J)$, then $\mathbb{D}_c^{\alpha}I^{\alpha}y(t)=y(t)$,
2) if $y\in AC^{1}(J)$, then

$$I^{\alpha c}\mathbb{D}^{\alpha}y(t)=y(t)-y(0)-y'(0)t.$$

then $y\in AC^{1}(J)$ involves finding a solution to the problem of boundary values

$$\mathbb{D}_c^{\alpha}y(t)+\mu(t)=0,\ t\in J,\ \ y(0)=y_0,\ \ y(1)=y_1, \qquad (13)$$

only in the event that y fulfills:

$$y(t)=\int_0^1 G(t,z)\mu(z)\mathrm{d}z+(y_1-y_0)t+y_0,\ \ t\in J, \qquad (14)$$

where the Green's function that is connected with (1) is defined by that which

$$G(t,z)=\frac{1}{\Gamma(\alpha)}\begin{cases} t(1-z)^{\alpha-1}-(t-z)^{\alpha-1}, & 0\le z\le t\le 1 \\ t(1-z)^{\alpha}, & 0\le z\le t\le 1 \end{cases} \quad (15)$$

Lemma 2 makes it possible for us to derive, based on equation (13), that

$$0=I^{\alpha c}\mathbb{D}^{\alpha}y(t)+I^{\alpha}\mu(t)=y(t)-y'(0)+\int_0^t\frac{(t-z)^{\alpha-1}}{\Gamma(\alpha)}\mu(z)\mathrm{d}z,$$

Moreover, the boundary conditions are present:

$$y(0)=y_0,\ y'(0)=y_1-y_0+\int_0^t\frac{(t-z)^{\alpha-1}}{\Gamma(\alpha)}\mu(z)\mathrm{d}z,$$

We are able to conclude that:

$$\begin{aligned} y(t)&=-\int_0^t\frac{(t-z)^{\alpha-1}}{\Gamma(\alpha)}\mu(z)dz+t\int_0^t\frac{(t-z)^{\alpha-1}}{\Gamma(\alpha)}\mu(z)\mathrm{d}z+(y_1-y_0)t+y_0 \\ &=\int_0^t G(t,z)\mu(z)\mathrm{d}z+(y_1-y_0)t+y_0,\ t\in J. \end{aligned}$$

Inversely, if $y\in AC^1(J)$ if (2) is satisfied, then we also have:

$$y(t)=-I^{\alpha}\mu(t)+\left(\int_0^t\frac{(t-z)^{\alpha-1}}{\Gamma(\alpha)}\mu(z)\mathrm{d}z+y_1-y_0\right)t+y_0.$$

(which shows that $\mathbb{D}_c^{\alpha}t=\mathbb{D}_c^{\alpha}1=0$) we have:

$$\mathbb{D}_c^{\alpha}y(t)=-\mu(t),\ t\in J,$$

that is, y is a proof of (1) [14].

Definition 1 If $\alpha\in\mathbb{R}$, Consequently, the fractional derivative αD is present at virtually every location on $\mathbb{D}$. If $\alpha>0$ If the value in question is not an integer, then it can be written as:

$$\left(\mathbb{D}_c^{\alpha}\mu\right)(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-z)^{n-\alpha-1}\mu^{(n)}(z)\mathrm{d}z \quad (16)$$

where $n=[\alpha]+1$ and $[\alpha]$ represents the greatest integer that is not necessarily bigger than α. Let us make advantage of this method in order to expand the idea of derivative to order that is not between integers; Take into consideration the nth power function derivation $g(x)$.

$$g(x)=x^k,\ x\ge 0 \quad (17)$$

$$\frac{d^n}{dx^n} g(x) = \frac{k!}{(k-n)!} x^{k-n} = \frac{\Gamma(1+k)}{\Gamma(1+k-n)} x^{k-n} \qquad (18)$$

In cases when both k and n are real integers, and where k n ≥, it may be possible to extend n to a real value represented by the letter in order to make the equation shown above more generalizable. n α :

$$\frac{d^\alpha}{dx^\alpha} g(x) = \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)} x^{k-\alpha} \qquad \mathbf{(19)}$$

The expansion of a function into a power series of x is the starting point for finding its fractional derivative; from there, one may apply equation (18) to derive the expansion. By using this method, you may get the fractional derivative of any function. As an example, consider the situation of derivatives $f(x) = e^{kx}$ rewrite, we are in order $f(x)$ according to the following:

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (20)$$

Hence

$$\begin{aligned}\frac{d^\alpha}{dx^\alpha} f(x) &= \frac{1}{\Gamma(1-\alpha)} x^{-\alpha} + \frac{1}{\Gamma(2-\alpha)} x^{1-\alpha} + \frac{1}{\Gamma(3-\alpha)} x^{3-\alpha} + \cdots \\ &= sign(x)\left(sign(x)k\right)^\alpha e^{kx}\left(1 - \frac{\Gamma(-\alpha, kx)}{\Gamma(2\alpha)}\right)\end{aligned} \qquad (21)$$

While Liouville found the fractional derivative of the exponential function in 1832, Riemann obtained the fractional derivative of the power function in 1847. Both of these derivatives were obtained by Liouville.

Riemann-Liouville Fractional Derivative

The idea of a fractional derivative operator of the Riemann-Liouville type will be applied here. To do this, we will examine the following approximation of the v-order Riemann Liouville fractional derivative of f (z):

$$\mathbb{D}_z^v\{f(z)\} = \int_0^z (z-t)^{-v-1} f(t)\,dt, \ \mathcal{R}(v) < 0 \qquad (23)$$

in which case the line connecting zero and z on the complex t-plane is the integration path.

A fractional derivative of by Riemann-Liouville of $f(z)$ of order v is defined as:

$$\mathbb{D}_z^{v}\{f(z)\} = \frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathbb{D}_z^{v-m}\{f(z)\}$$
$$= \frac{\mathrm{d}^m}{\mathrm{d}z^m}\left\{\frac{1}{\Gamma(-v+m)}\int_0^z (z-t)^{-v+m-1} f(t)\mathrm{d}t\right\}, \mathcal{R}(v) < 0 \qquad (24)$$

When $\mathcal{R}(v)$, let $m \in \mathbb{N}$ be the least positive integer that is more than $\mathcal{R}(v)$ and so $m-1 \le \mathcal{R}(v) < m$, after that, a novel fractional derivative of Riemann-Liouville $f(z)$ can be described in the following way for orders v:

$$\mathbb{D}_z^{v,\{p\}_q}\{f(z)\} = \frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathbb{D}_z^{v-m,\{p\}_q}\{f(z)\}$$
$$= \frac{\mathrm{d}^m}{\mathrm{d}z^m}\left\{\frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-v+m)}\int_0^z f(t)(z-t)^{-v+m-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right)\mathrm{d}t\right\}, \qquad (25)$$
$$(\Re(p) > 0, \Re(q) > 0)$$

Remark On setting $p = 0, q = 0$ The traditional Riemann-Liouville fractional derivative remains in equations (18) and (19). The well-known fractional derivative operator is reduced to in Equations (24) and (25) when q = 0.\

Spectral Methods

Quantitative methods that exhibit spectrum strategies for solving differential equations are quite accurate. A finite linear combination of a specified group of orthogonal basis functions is used to define the numerical solution, as per Boyd (2000) {φk(x)}∞ k=0, as

$$f_n(x) = \sum_{k=0}^{n} \widehat{f}_k \phi_k(x) \,,$$

where $\widehat{f}_k$ The items in issue are the s spectral coefficients of f with regard to the base. $\{\phi_k\}$.

$\widehat{f}_k$ Is a measurement that specifies the degree to which f is oriented in the direction of ϕ_k as well as assessed by

$$\widehat{f}_k = \int_a^b f(x)\phi_k(x)w(x)\mathrm{d}x \,,$$

while simultaneously being related to the basis is the weight function w(x) $\{\phi_k\}$, and the range [a, b] alike. An approximation of f's derivative can be made using f_n' which is given by

$$f'_n(x) = \sum_{k=0}^{n} \widehat{f}_k \phi'(x).$$

There are three main types of spectral approaches: Galerkin, Tau, and collocation. In equation, the statement f 0 n (x) represents the Galerkin spectral technique. The spectral coefficients are used to express the derivative of f, which is why this occurs. Tau spectral technique differs from Galerkin spectral approach in that it may satisfy boundary conditions without the basis function. The sole distinction between the two approaches is this. Since both the collocation spectral technique and the finite difference method employ a set of grid points called collocation points, this is why they are similar. Collocation relies on interpolation as its base. We shall focus on collocation techniques in the future essay.

Our collocation spectral method will make use of Chebyshev points.

Chebyshev Points.

Through the employment of Chebyshev points, it is possible to achieve a discretization of both FDEs and ODEs respectively. Our explanation of the Chebyshev points that we have is based on the information that can be found in Trefethen (2000). All of these arguments are, in fact,

$$x_j = -\cos\left(\frac{j\pi}{n}\right), \quad j = 0, 1, \ldots, n\,,$$

In the process of spectral collocation, it is common practice to make use of the extrema of Tn(x) on the interval [−1, 1]. Give some thought to the equation x = -cos θ. Assuming that we divide the variable θ, which is a member of the interval [0, π], in an even way on the upper half plane of a unit circle, as seen in Figure 1, let us proceed with the following assumption.

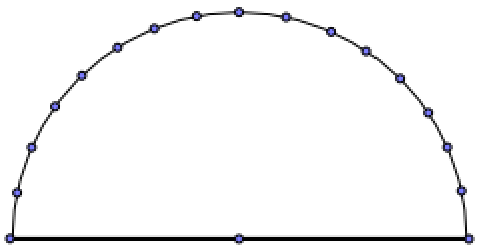

Figure 1 : The interval lies on the upper half plane of a unit circle, where the angle θ is evenly divided. **[0, π].**

Projecting these points on the X-axis [−1, 1], we obtain Figure 2.

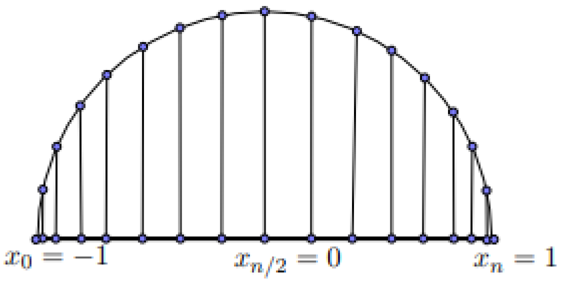

Figure 2: The projections of the split angle are created from the circle's boundary to the x-axis.

Figure 2 makes it very obvious that the spots cluster toward the edges. Among the many advantages of Chebyshev points is the fact that they allow for a finer boundary resolution. Using these grouped points allows the polynomial interpolant to significantly enhance its accuracy. As an example, we create an interpolation polynomial for this parameter $\frac{1}{1+25x^2}$ Figure 3 shows that this occurs at both equispaced and Chebyshev points.

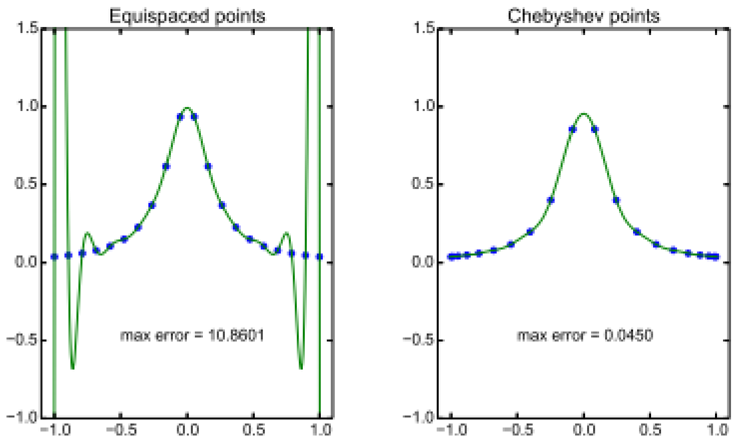

Figure 3: Interpolation is a procedure A polynomial that shows how using clustered points in the analysis impacts the results.

When compared to the equispaced points, which perform quite badly, the Chebyshev points stand out beautifully in Figure 3. Chebyshev points, the foundation of our spectral system, will be covered in the next paper.

Tempered fractional advection with d = 0.

To illustrate We begin by thinking about the exact response as

$$u(x) = e^{-x} \cdot {}_{-1}I_x^{\frac{\alpha_1}{2}} v(x) = \begin{cases} -\frac{(x+1)^{\frac{\alpha_1}{2}}}{\Gamma(1+\frac{\alpha_1}{2})}, & x \in [-1, 0], \\ \frac{2x^{\frac{\alpha_1}{2}} - (x+1)^{\frac{\alpha_1}{2}}}{\Gamma(1+\frac{\alpha_1}{2})}, & x \in (0, 1], \end{cases}$$

and

$$f(x) = \begin{cases} -\frac{(x+1)^{-\frac{\alpha_1}{2}}}{\Gamma\left(1-\frac{\alpha_1}{2}\right)} \cdot e^{-x}, & x \in [-1, 0], \\ \frac{2x^{-\frac{\alpha_1}{2}} - (x+1)^{-\frac{\alpha_1}{2}}}{\Gamma\left(1-\frac{\alpha_1}{2}\right)} \cdot e^{-x}, & x \in (0, 1]. \end{cases}$$

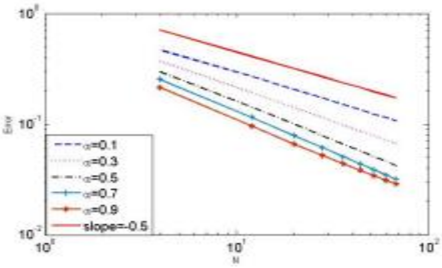

Figure 4. Here are the numerical L2 errors for different α1 of the problem and its real solution:

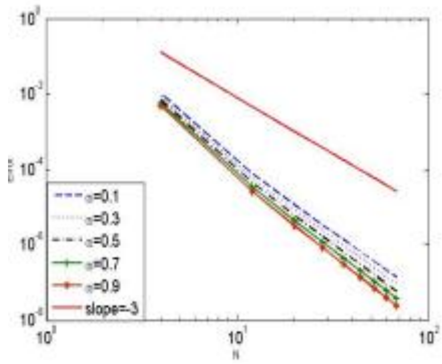

Figure 5. Based on the correct answer, the numerical L2 errors for different α1 of the problem are shown below.

Here

$$v(x) = \begin{cases} -1, & x \in [-1, 0], \\ 1, & x \in (0, 1], \end{cases}$$

belongs to L2(−1, 1).

Figure 4 confirms what was predicted in Remark 3.16: errors in the L2(−1, 1) sense decrease, even when N −1/2 is considered. The expected behavior has been verified. As the α1 value increases, we can see that the error decreases.

Examples Following this, we will consider the exact answer as follows:

$$u(x) = e^{-x} \cdot {}_{-1}I_x^{\frac{\alpha_1}{2}} v(x) = \begin{cases} -\frac{(x+1)^{3+\frac{\alpha_1}{2}}}{\Gamma\left(4+\frac{\alpha_1}{2}\right)}, & x \in [-1,0], \\ \frac{2x^{3+\frac{\alpha_1}{2}} - (x+1)^{3+\frac{\alpha_1}{2}}}{\Gamma\left(4+\frac{\alpha_1}{2}\right)}, & x \in (0,1], \end{cases}$$

and

$$f(x) = \begin{cases} -\frac{(x+1)^{3-\frac{\alpha_1}{2}}}{\Gamma\left(4-\frac{\alpha_1}{2}\right)} \cdot e^{-x}, & x \in [-1,0], \\ \frac{2x^{3-\frac{\alpha_1}{2}} - (x+1)^{3-\frac{\alpha_1}{2}}}{\Gamma\left(4-\frac{\alpha_1}{2}\right)}, \cdot e^{-x} & x \in (0,1], \end{cases}$$

where

$$D^{(3)} v(x) = \begin{cases} -1, & x \in [-1,0], \\ 1, & x \in (0,1]. \end{cases}$$

Thus v(x) belongs to H^3 (−1, 1).

Figure 5 makes it very evident that the errors agree with the estimate in the Theorem.

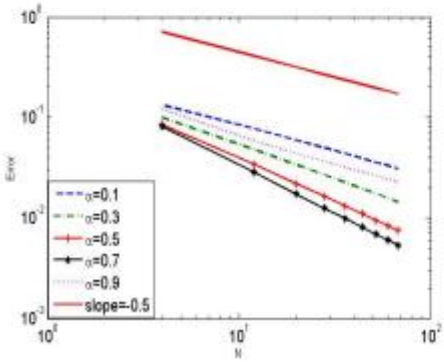

Figure 6. For various values of α1, the numerical L2 errors are calculated using the appropriate function as the basis.

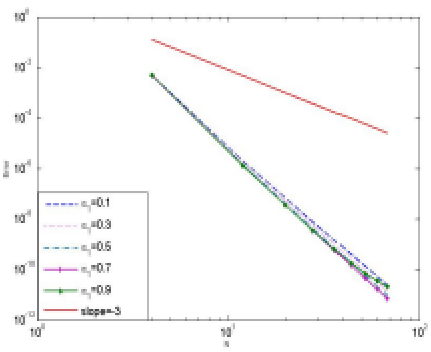

Figure 7. The numerical L2 errors for various values of α1 of the problem with α2 equal to zero, d equal to -500, and the true solution as where m equals three and γ equals three-thirds.

Conclusion

Fractional differential equations (FDEs) emerged as an effective extension of classical calculus and laid the ground for modeling complicated systems with memory, hereditary traits, and non-standard behavior. This research revealed the significance of FDEs across a number of fields; it showed that FDEs are capable of predicting real-world situations more efficiently than classical integer order models. FDEs come in handy, since they can account for memory and non-local effects. Examples in physics, biology and the economy, where FDEs were applied to solve problems of anomalous diffusion, dynamics of neural networks, and long-term correlations illustrate the importance of the equations. When it is time to solve FDEs, many numerical algorithms work very well. As an instance you have finite difference and finite element methods, and Adams-Bashforth-Moulton method. But calculating solutions to FDEs can be quite complicated when it comes to bigger and more complicated problems. Analytical techniques have also proven to be the ones for yielding the right or rather accurate solutions. They assisted biologists in improving their predictions concerning population dynamics and disease spread. When it comes to regulating systems, engineering's fractional controllers proved superior; when it comes to modeling fluctuations in finance, we should go for fractional differential equations. Generally speaking, fractional differential equations are an excellent mathematical resource for determining the causes and predicts of different complicated processes occurring in reality. Moreover, as stated earlier, since fractional differential equations offer a more comprehensive framework when compared to the conventional models, they may make a significant contribution to scientific and industrial progress.

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